Teaching Coupled Differential Equations by the Network Method

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ABSTRACT: *Much of effort is devoted by students to the numerical or approximated analytical solutions of the complex problems in which the mathematical model is integrated by coupled differential nonlineal transient equations. In particular, difficult convergence problems arising from numerical methods are inevitable. The use of network analogy, which requires very few devices and rules for circuit design, has been demonstrated to be very useful as regards the accuracy of its solutions, which are obtained by an appropriate educational software. All the mathematical requirements needed for the convergence are realized by the mentioned software, not by the programmer. Applications to the Lorenz and Loka-Volterra and problems are presented.*

1 INTRODUCTION

An important number of natural problems (physical, chemical and biological) are formulated by a mathematical model that contains coupled (conjugated) differential equations, whose solution is complicated for most of students of engineering. Conjugated problems are stated by a system of equations in which dependent variables appear in more than one. Such occurs, for example, in chemical diffusion-reaction systems, where the chemical components are continuously combining and simultaneously diffusing in the reactor, or in the transport of viscous fluids, where the temperature depends on the velocity gradients, or in ecological models, where the biological species interact very closely.

The solution to this kind of problem is by no means like trivial since, in general, no analytical (mathematical) solution exists and, in general, the equations of the mathematical model are nonlineal. In this way, the use of approximate numerical methods which, in turn, require a high degree of specialization far beyond the knowledge of undergraduate students, is indispensable. The network simulation method (NSM, hereinafter), widely used in a large variety of engineering problems (1,2,3), is an interesting, useful and easy to learn tool that may give help to students in solving this kind of problem.

NSM takes advantage of the commercial computer codes to analyse electric networks which involve sophisticated techniques for the numerical solution of great variety of differential equations appearing in engineering problems. The only requirement of NSM (2) is to design a network model, the equations of which are formally equivalent to those of the mathematical model (time remains as a continuous variable, while spatial variable is discretized). Nevertheless, this design is not a serious objection since very few devices are needed to implement in the network any kind of term (addends) of the differential equations regardless of its inherent nonlinearity (each term is connected to an only device). Besides, very few programming rules are required to elaborate the network file needed for running it in the PC. The rest of the work, as regards the accuracy and convergence problems, is made by the computer code. Pspice (4,5) is used in this work.

The common equivalence of these conjugated processes between the variables of the real problem and those of the network must take into account that the network potential must be a single-value continuous variable, whereas its gradient represents the instantaneous current flow. Examples: species concentration in ecological problems is connected to network potential, whereas the species flux is connected to the network current, and temperature is connected to network potential, whereas heat flux is connected to the network current.

The steps for designing the equivalent networks that solves these kinds of coupled nonlineal problems are set out in detail in this work. Starting from these rules, students can learn programming and experiment with their own designs. Applications to Lotka-Volterra and Lorenz problems are shown.

2 NOMENCLATURE

A_N, B_N, C_N , …	dimensionless or nondimensionless constants, N=1,2,3,…
c	species concentration (members m^{-3})
C	capacitor, also capacitance
I_C	network current
f, g, h	arbitrary mathematical functions
t	time (s)
V	network voltage
Subscripts:	
i, j, k	connected to the kind of species i, j, k, respectively
Superscripts:	
m, n, l,	real numbers

3 BASIC NETWORKS DEVICES AND MATHEMATICAL MODEL

The differential equations that take part in the mathematical model are formed by an algebraic sum of terms like these:

$A_o(dc_i/dt)$	temporal change of the dependent variable c_i (i=1,2, …),
B_o	dimensionless constant
$C_o\, c_i$	a fraction of the dependent variable c_i,
$D_o\, c_i^n$	a fraction of the polynomial function of the dependent variable c_i,
$E_o\, c_i^n\, c_j^m$	a fraction of the product of polynomial functions of two dependent variables, c_i and c_j,
$F_o\, c_i^n\, c_j^m\, c_k^l$	a fraction of the product of polynomial functions of three dependent variables, c_i, c_j and c_k,
$f(c_i)$	an arbitrary function of the dependent variable c_i,
$g(c_i,c_j)$	an arbitrary function of the three dependent variables c_i and c_j,
$h(c_i,c_j,c_k)$	an arbitrary function of the two dependent variables c_i, c_j and c_k,

According to Kirchhoffs current law, we may consider that each term defines an electric current that flows inside or outside, depending on the sign, to the same point in the network model. As regards the conservation of the network flux and the uniqueness of the network potential at any instant, no complementary condition is needed since the circuit laws (Kirshhoff currents and voltage laws), which are immediately satisfied, assume these requirements.

The first term, $A_o(dc_i/dt)$, is implemented in the model by a capacitor. As is well known, the electric current through a capacitor, i_C, depends on the electric potential at its ends, V, in the form, $i_C = C((dV/dt)$, where C is the capacitance. Hence, the term $A_o(dc_i/dt)$ is assumed by a capacitor of capacitance A_o. The initial voltage at the capacitors may be specified separately and corresponds to the initial condition V(t=0).

The rest of the terms are implemented by the same device, named “controlled current source”. This is a special (very versatile) device that provides a current that may by specified by software. The current may be an arbitrary function of one or more potential (dependent variable) values. For each time, the concentration values are read at the points of the model where they are found, and operated adequately into the source to provide the required current output. For example, the obtain the current $E_o c_i^n\, c_j^m$ at any instant, the source reads the concentration values c_i and c_j (in the points of the network where they are defined), and works out the value of $E_o\, c_i^n\, c_j^m$. This value is the output current of the source at that instant.

The kind of problem we deal with, consisting of three dependent variables for simplicity, is formulated by a system of differential equations in the form

$$dc_1/dt = A_1 + B_1c_1 + C_1c_2 + D_1c_3 + E_1c_1c_1 + F_1c_1c_2 + G_1c_1c_3 + H_1c_2c_3 + I_1c_1c_2c_3$$

$$dc_2/dt = A_2 + B_2c_1 + C_2c_2 + D_2c_3 + E_2c_2c_2 + F_2c_1c_2 + G_1c_1c_3 + H_2c_2c_3 + I_2c_1c_2c_3$$

$$dc_3/dt = A_3 + B_3c_1 + C_3c_2 + D_3c_3 + E_3c_3c_3 + F_3c_1c_2 + G_1c_1c_3 + H_3c_2c_3 + I_3c_1c_2c_3$$

The concentrations c_i on the right of the above equations may be polynomial functions of c_i. (for simplicity, these functions have been omitted). The mathematical model is completed by assuming initial conditions, this is

$$c_{1(t=0)} = c_{1,0},\ \ c_{2(t=0)} = c_{2,0},\ \ c_{3(t=0)} = c_{3,0},$$

4 APPLICATIONS

The Lorenz attractor. As the first application of the proposed method let us solve the Lorenz attractor´s problem discovered in 1962 (6), the first strange attractor ever recognized as such in the natural sciences and for which the answers to some very basic questions are still to be found. The system under study, initially formed by a 7-equation model (7), deals with the convective motion in a fluid heated from below and cooled from above to study the chaotic behaviour of the atmospheric layers. Guidelines to interpret this behaviour are sought. Lorenz simplified the above Navier-Stokes 7-eq.model to a system of three ordinary differential equations:

$$dc_1/dt = A_1(c_2 - c_1)$$

$$dc_2/dt = - c_2 + B_1c_1 - c_1c_3$$

$$dc_3/dt = - C_1c_3 + c_1c_2$$

In this idealized model, the warm fluid below rises and the cool fluid above sinks, setting up a clockwise or counterclockwise current. c_1 represents the fluid flow velocity, c_2 is the temperature difference between ascending and descending fluid and c_3 is the distortion of the vertical temperature profile from its equilibrium. The parameters of the system A_1, and B_2 represent the dimensionless Prandtl (ratio between the kinetic viscosity and thermal diffusivity) and Reynolds (ratio between inertial and viscous forces) numbers, while C_1 is a form factor of the recipient (width/height ratio) where the gases are confined.

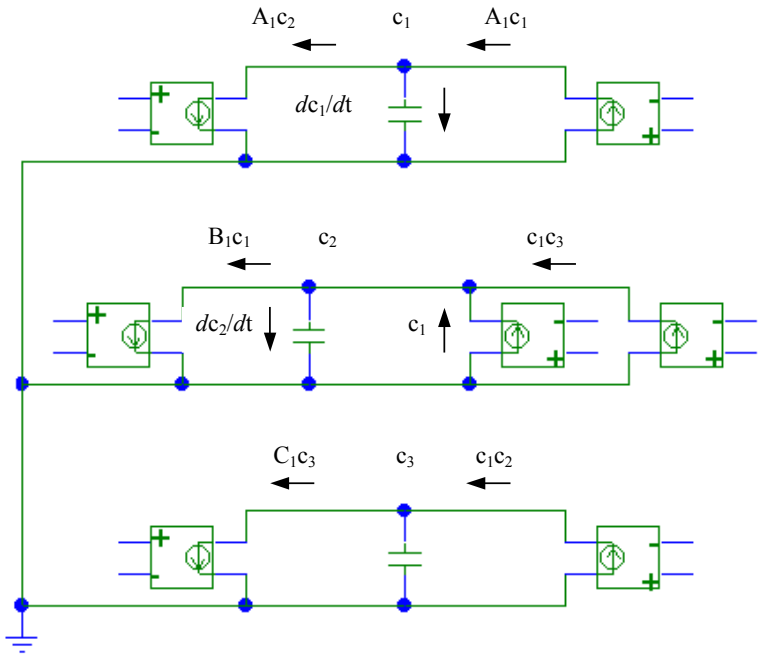

Figure 1. Network model of Lorenz attractor

The network model is shown in figure 1. It contains three separate networks, one for each equation. The current sense of the controlled current sources in each network agrees with the sign of the equation. Numerical values applied to the problem are: A_1=10, B_1=28, C_1=8/3, $c_{1(t=0)}$= $c_{2(t=0)}$=$c_{3(t=0)}$=10, Time interval = 100 s. After numbering the nodes of the network the text file to be run in Ppsice is:

```
Cx 1 0 1 IC=10
Gxsal 1 0 VALUE={10*V(1,0)}
Gxent 1 0 VALUE={-10*V(2,0)}
Cy 2 0 1 IC=10
Gysal1 2 0 VALUE={V(3,0)*V(1,0)}
Gysal2 2 0 VALUE={V(2,0)}
Gyent 2 0 VALUE={-28*V(1,0)}
Cz 3 0 1 IC=10
Gzsal 3 0 VALUE={8*V(3,0)/3}
Gzent 3 0 VALUE={-V(1,0)*V(2,0)}
.TRAN 0 100 UIC
.OPTIONS NOPAGE RELTOL=.01
.END
```

The five last sentences allude to the time interval to be simulated and to the numerical accuracy of the solution. Time computing in a PC less than 1 s.

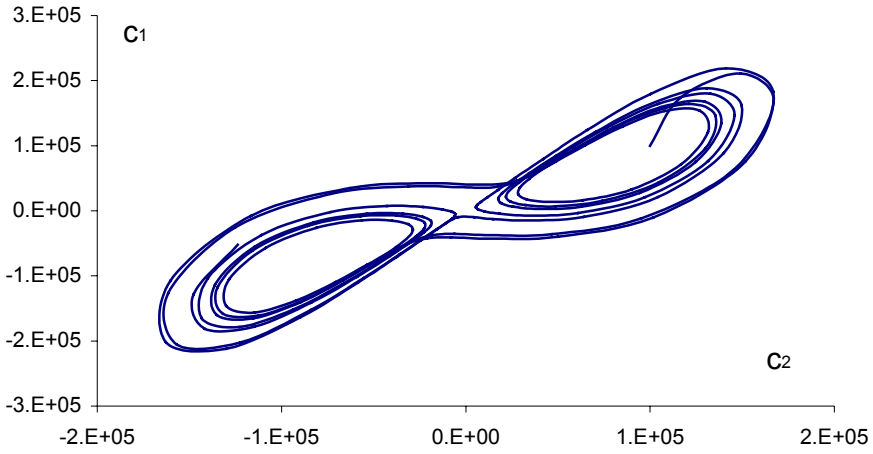

Figure 2. Phase Diagram of Lorenz problem, c_1-c_2

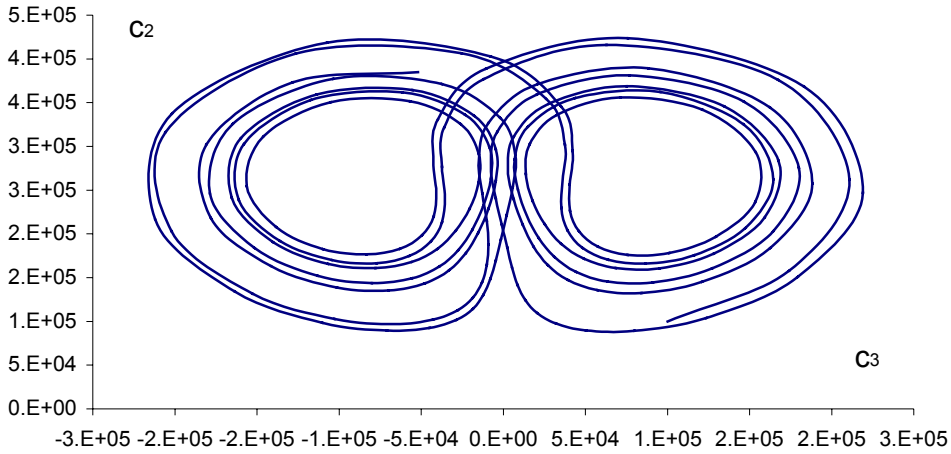

Figure 3. Phase Diagram of Lorenz problem, c_2-c_3

Figures 2 and 3, c_1 as a function of c_2 and c_2 as a function of c_3, respectively, show the solution of the problem (typical phase diagrams of the attractor).

The Lotka-Volterra problem. A family of models called the Lotka-Volterra equations (8) are often used to simulate interactions between two or ore populations. The three competition species Lotka-Volterra model is form by the system of ordinary differential equations

$$dc_1/dt = A_1c_1 - B_1c_1c_2 - C_1c_1c_3$$

$$dc_2/dt = A_2c_2 + B_2c_1c_2 - C_2c_2c_3$$

$$dc_3/dt = -A_3c_3 + B_3c_1c_3 + C_3c_2c_3$$

$c_{i(I=1,2,3)}$ is the time dependent species densities. The network model is shown in figure 4. Numerical values are: A_1=50, A_3=10, A_2=B_1=C_1=B_2=C_2=B_3=C_3=1, $c_{1(t=0)}$=50, $c_{2(t=0)}$=$c_{3(t=0)}$=10, Time interval ≈500 s.

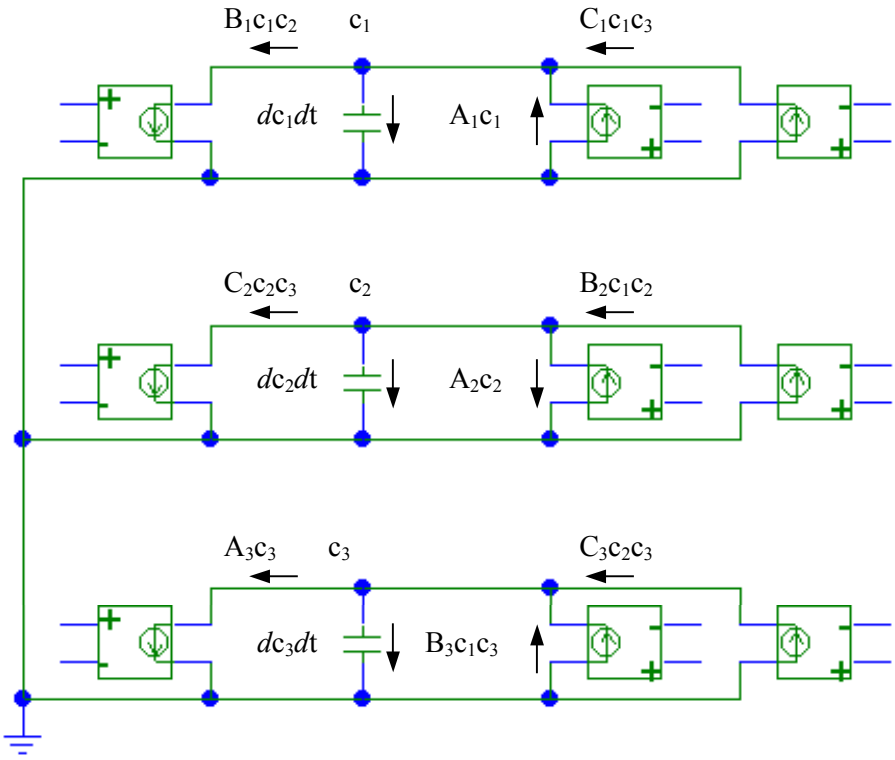

Figure 4. Network model of Lotka-Volterra problem

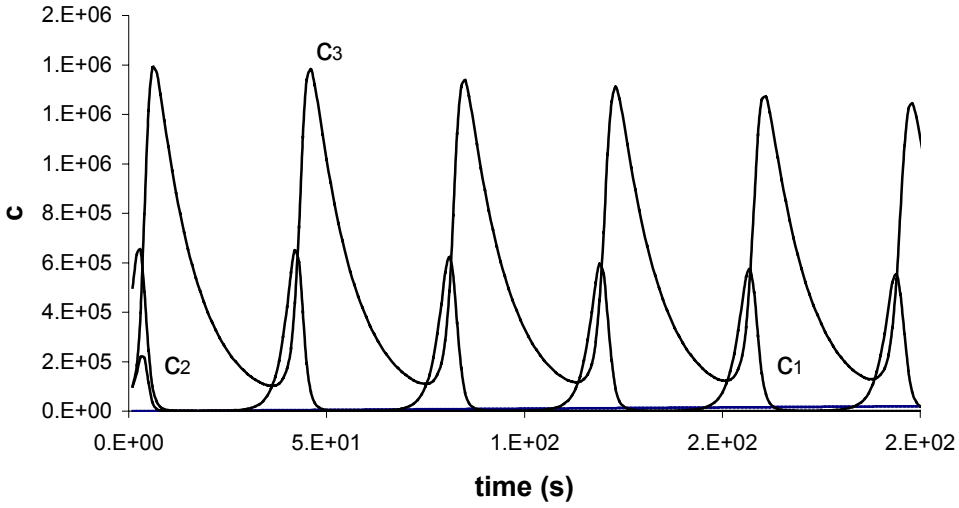

Figure 5. Time evolution of species concentration

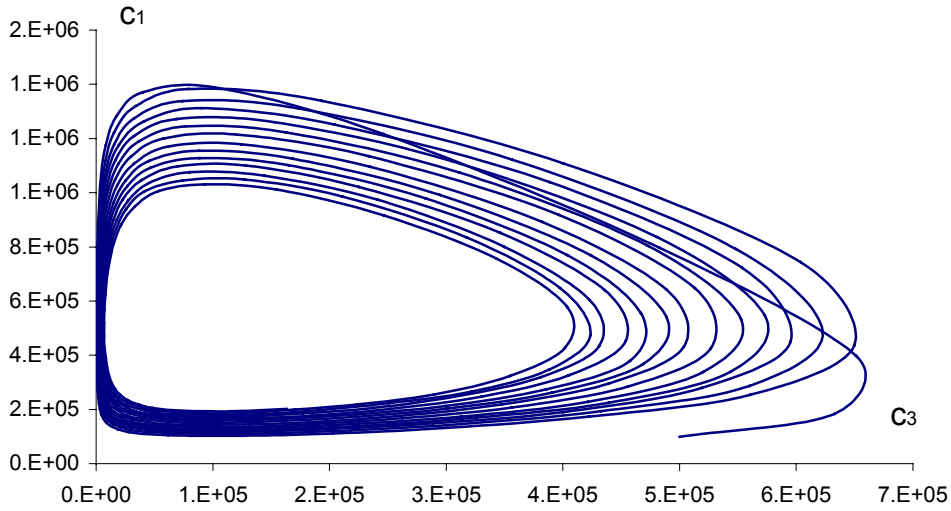

Figure 6. Phase Diagram of Lotka-Volterra problem, c_1-c_3

Figure 5 shows the temporal evolution of species concentration; one of the species (c_2) disappears at the beginning. Figure 6 depicts c_1 as a function of c_3, sowing a cyclic structure in its evolution. Again, time computing in a PC less than 1 s.

5 CONCLUSIONS

A simple, versatile and accurate model based on the network simulation method has been proposed for the numerical solution of the problems whose mathematical model includes coupled differential equations. This is a kind of non-trivial problem whose solution requires the use of numerical techniques. The design of the network model uses very few programming rules since only two devices are required whatever be the form of the nonlineal term in the differential equation. Thanks to this facility the student, with a minimum effort is able to design the model and to elaborate the pspice file to be running. Typical examples are included in the text to demonstrate the power of the proposed method.

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