The Teaching of Unsteady Heat Conduction Using the Thermo-electric Analogy and the Code pspice. Nonlineal Models

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ABSTRACT: *Differential and finite equations of lineal electric and heat transfer processes are equivalent and many textbooks devote long paragraphs to this important aspect for the education of mechanical engineers. Nevertheless, no reference alludes to nonlineal differential equations which, in practice, better simulate the real processes.*

In this paper, the basic electrical devises incorporated in most of the educational software simulation circuits, which may be used to implement the nonlineal terms of these differential equations, are presented. Once the equivalent network is designed the main advantage is that the numerical solution of the problem is immediately obtained by the code pspice without later mathematical manipulations. Applications are presented for the heat transport (conduction) transient equations in solids with thermal properties dependent on the temperature.

1 INTRODUCTION

As is well known, the mathematical models behind the dynamic behaviour from different disciplines of physical systems have much in common. For example, the partial differential equation of transient diffusion processes is the same for: i) the transport of water in soils in the field of agricultural science, ii) the transport of strange atoms in pure semiconductors in the field of solid state electronic, iii) heat conduction in solids, and so on. In consequence the solutions for all these problems are essentially the same and generally are formed by complicated infinite series that diverge markedly for short times and that require a high grade of mathematical manipulation (Carslaw, 1980 and Özisik, 1993). The method of separation of variables is cited as the primacy procedure for solving lineal diffusion problems in regular bodies.

Solving differential equations in which the time is the only independent variable using electric analogical circuits, particularly operational devices, is an extended technique known by all students of electrical engineering courses. Nevertheless, when the problem contains partial differential equations with space and time as independent variables, as in the case of the diffusion equations studied here, the solution is much more complicated.

Two steps (González-Fernández and Alhama, 2002 and Alhama, 1999) are needed: on the one hand, the spatial variable of the mathematical model is discretized resulting in a set of finite difference differential equations with time as a continuous variable; this set is used to design the whole network model; on the other hand, simulation of this network is run in a suitable computer code, Pspice being the code used in this work (Nagel, 1977 and Pspice, 1994).

In heat conduction thermodynamic systems in particular, it is widely known that all textbooks (Incropera, 1996 and Mills, 1995) incorporate the electric circuit analogy, albeit on a limited basis (lineal problems). Historically, this analogy comes from the experimental works of Paschkis and Baker (1942) and Paschkiss and Heisler (1944) with their heat and mass analyser, a laboratory rack composed of resistors and capacitors.

Using the analogies between the pairs of quantities

Electric voltage, V, (V) « Temperature, T, (K)

Electric current, i, (A) « Heat flux density, j, (Wm^{-2})

the effect of thermal conductivity and heat capacity may be implemented in the network by the lineal devices, electric resistor and electric capacitor, respectively (Campo and Alhama, 2003). As regards any nonlineal terms contained in the differential equation of the mathematical models, simple modern devices such as "controlled voltage sources" and "controlled current sources" permit the implementation of any kind of nonlineality by specifying the control of these devices using a few simple programming rules. As regards the (lineal or nonlineal) boundary conditions, these may also be easily implemented by the mentioned controlled sources. Once all the devices have been implemented, simulation is carried out by means of a circuit simulation computer code (pspice).

The central objective of this paper on engineering education is to provide students with an alternative, simple and rapid computer method, based on the network simulation method, that enables them to obtain accurate numerical solutions to heat conduction lineal and nonlineal problems.

2 THE MATHEMATICAL MODEL

Taking into account Fourier´s law that connects the heat flux density with the temperature gradient, "$\mathbf{j} = -k\ \nabla(T)$", the local heat balance equation is given by equation (1a), which is the general heat conduction equation for heterogeneous anisotropy solids. This equation may be considered as Kirchhoff´s current law, T being a continuous and single-valued variable. If the solid is isotropic the thermal conductivity is a scalar, k, and equation (1a) reduces to equation (1b), where ∇^2 is the Laplacian operator. If the solid is homogeneous (k independent of the spatial coordinates), equation (1b) simplifies to equation (1c).

Furthermore, one or the two last terms of equation (1c) may be zero. If no heat sources or sinks exist, $\sigma = 0$, so that equation (1c) reduces to (1d), which is properly called the heat diffusion equation and if, steady state is considered, (1d) may be written as equation (1e), called the Poisson equation, which can be even more simplified if $\sigma = 0$, resulting in Laplace equation, (1f).

Besides the local balance equation, temperature dependencies of thermophysical properties k, and c_e for nonlineal solid are required; these are given by functions of the form of equations (2) and (3). Finally, boundary and initial conditions are also needed for the final solution of the problem. Boundary conditions, which may be either lineal or nonlineal, give information about the temperature or heat flux at the extremes of the solid; this specification is connected to either of the following expressions according to the type of condition, whereas the initial condition, equation (5), informs us of the solid temperature before the process begins.

$$\rho\ c_e(\partial T/\partial t) - \nabla\cdot(\kappa\ \nabla(T)) - \sigma = 0 \qquad (1a)$$

$$\rho\ c_e(\partial T/\partial t) - k\nabla^2(T) - \nabla k\cdot\nabla(T) - \sigma = 0 \qquad (1b)$$

$$\alpha^{-1}\ (\partial T/\partial t) - \nabla^2(T) - \sigma/k = 0 \qquad (1c)$$

$$\alpha^{-1}\ (\partial T/\partial t) - \nabla^2(T) = 0 \qquad (1d)$$

$$\nabla^2(T) + \sigma/k = 0 \qquad (1e)$$

$$\nabla^2(T) + \sigma/k = 0 \qquad (1f)$$

$$k = k(T) \qquad (2)$$

$$c_e = c_e(T) \qquad (3)$$

$$T_{ext} = T(t),\ \text{or}\ j_{ext} = j(T),\ \text{or}\ j_{ext} = j(T_{ext}) \qquad (4)$$

$$T_{ini} = T_o \qquad (5)$$

The mathematical model is formed by the set of equations (1-5).

3 THE NETWORK MODEL

Assuming, for example, spherical coordinates, the first step in designing a network () model of a control volume is to discretize the spatial variable, r, of equation (1a-e). To this end, a cell of thickness Δr_i and radii $r_{i-\Delta}$, r_i and $r_{i+\Delta}$ is considered (figure 1a). The finite difference equation, assuming the nomenclature of figure 1b, can be written in the form

$$\Gamma_i\, \rho c_{e,i}(dT_i/dt) = [S_{i-\Delta}\, k(T_{i-\Delta} - T_i)/(\Delta r_i/2)] - [S_{i+\Delta}\, k(T_i - T_{i+\Delta})/(\Delta r_i\,/2)] \qquad (6)$$

where $S_{i-\Delta}$ and $S_{i+\Delta}$ are the surfaces of radii $r_{i-\Delta}$ and $r_{i+\Delta}$, respectively, and Γ_i the volume of the control volume.

Writing equation (6) in the form $j_{i,\gamma}=j_{i-\Delta}-j_{i+\Delta}$, where $j_{i,\gamma} = \Gamma i\, \rho c_{e,i}(dT_i/dt)$, $j_{i-\Delta} = S_{i-\Delta}\, k(T_{i-\Delta} - T_i)/(\Delta r_i/2)$ and $j_{i+\Delta}=S_{i+\Delta}k(T_i-T_{i+\Delta})/(\Delta r_i/2)$, each of these equations for lineal solids (k and c_e constants) defines a capacitor and two resistors, respectively,of values $C_i=\Gamma i\, \rho c_{e,i}$, $R_{i-\Delta}=(\Delta r_i)(2S_{i-\Delta}\, k)^{-1}$ and $R_{i+\Delta}=(\Delta r_i)/(2S_{i+\Delta}k)^{-1}$ which are electrically connected as in figure 2. N network cells are connected in series to make up the whole medium.

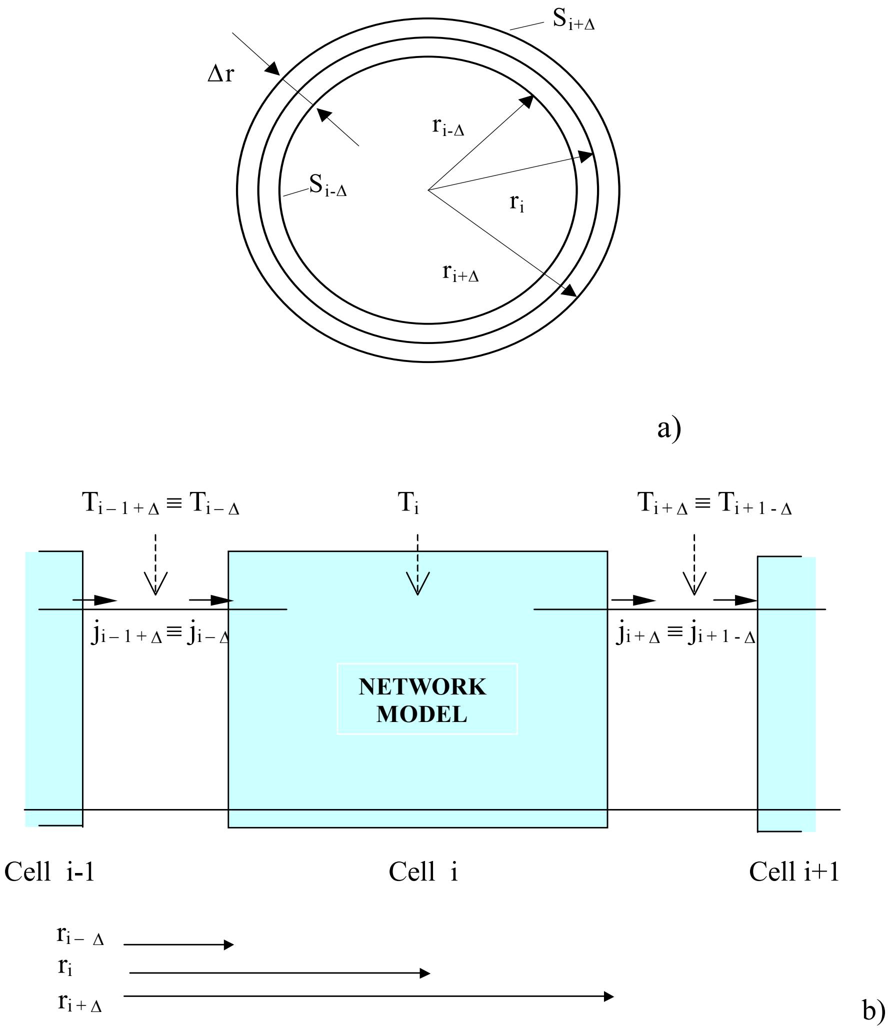

Figure 1 – Geometry and nomenclature for the cell i

As regards nonlineal solids equations, (2) and (3) must be implemented. To this end, two values for the conductivity are defined, one on the left of the cell, $k_{i-\Delta}$ (controlled by the temperature at that end, $T_{i-\Delta}$) and $k_{i+\Delta}$ (controlled by $T_{i+\Delta}$); that is $k_{i\pm\Delta}= k(T_{i\pm\Delta})$; on the other hand, the specific heat, $c_{e,i}$ is evaluated by means of the temperature T_i at the centre of the cell, so $c_{e,i} = c_{e,i}(T_i)$. The network model for the cell is now designed making use of certain non-lineal devices, called “control current-generators”, that are defined within the library of the commercial computer code Pspice. These sources, namely G in the figure 3, provide output currents (defined by software) able to include any kind of nonlineal functions of equations (2) and (3). In this way, only five devices are needed to design the network model in the case of the 2-D orthotropic diffusion problem, figure 3.

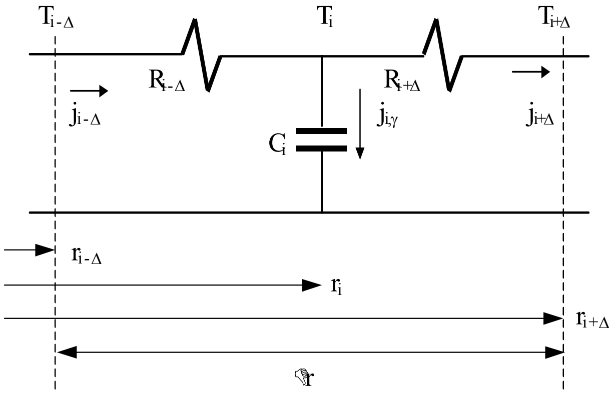

Figure 2 – 1-D model for the lineal solid

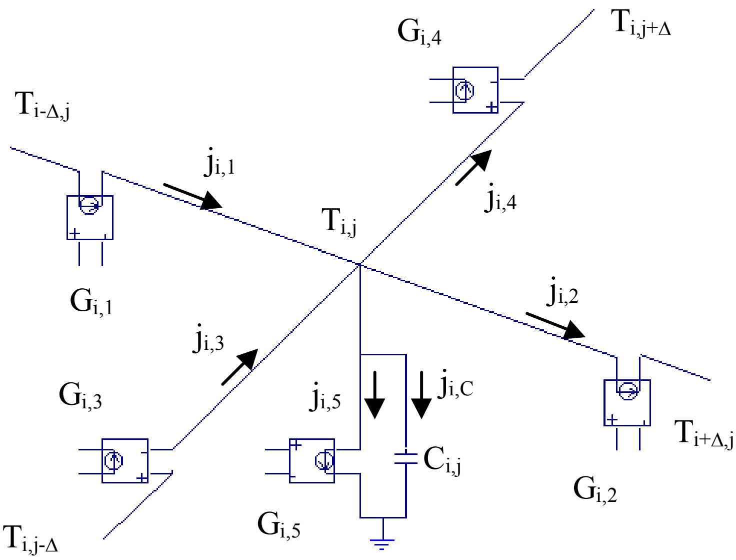

Figure 3 – 2-D model for the nonlineal solid

Boundary conditions on any kind (constant temperature or heat flux, convective, radiative or a compatible combination of these conditions) may be implemented in the network by means of the mentioned controlled current or voltage sources. Finally, initial condition is applied by fixing the initial charge in the condensers.

4 APPLICATIONS

Simple numbers been adopted for the numerical values of the quantities in the applications have since the main subject of this work is to show the power of the method although the nonlineal effects may also be appreciated.

First application. A flat rectangular orthotropic plate (figure 4), placed in contact with air is warned up through the half upper side and cold down slower through the downer, both by convection. The rest of its bounds are adiabatic. Numerical values of the problem are:

$L_x = 1$, $L_y = 0.25$, $N_x = 20$, $N_y = 5$, $\Delta x = \Delta y = 0.05$,
$k_x = k_{x,0} + k_{x,1}$ (T-273) = 1 + 1(T-273), $k_y = k_{y,0} + k_{y,1}$ (T-273) = (1/6) + 0.1(T-273), ρ=1, $c_e = 400$,
$T_{ini} = 273$, $T_{ref,c,up} = 274$, $h_{up} = 10$, $T_{ref,c,down} = 273$, $h_{down} = 1$,
$j(0,y) = j(L_x,y) = 0$,
$j(0<x<1,0) = j_2 = h_{down}\,[T(x,0) - T_{ref,c,down}] = 0.1[T(x,0)-273]$,
$j(0<x<0.5,L_y) = 0$, $j(0.5<x<1,L_y) = j_1 = h_{up}\,[T_{ref,c,up} - T(x,L_y)] = 10\,[274- T(x,L_y)]$

With these values, $R_{i\pm\Delta,j} = 0.025$, $R_{i,\,j\pm\Delta} = 1.5$, $C_{i,j} = 1$. Figure 5 depicts the nonsteady temperatures at five typical locations (x,y) on the plate for the lineal case, 1: (0.5,0.25), 2: (0.75,0.25), 3:(=1,0.15), 4:(0,0.125), 5:(0.5,0), while the same temperatures for the nonlineal case are depicted in figure 6. The effect of the increasing the temperature in the second case may be appreciated. The fluxes of warming and cooling are represented in figure 7 for both lineal and nonlineal cases. Again, the influence of nonlinearity is shown. Time computing is less than 2.2 s. for both cases.

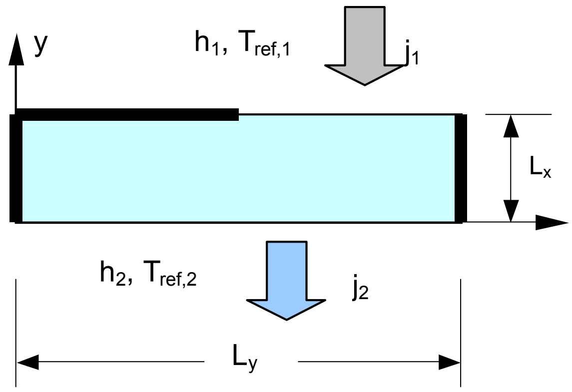

Figure 4 – Geometry of the first application

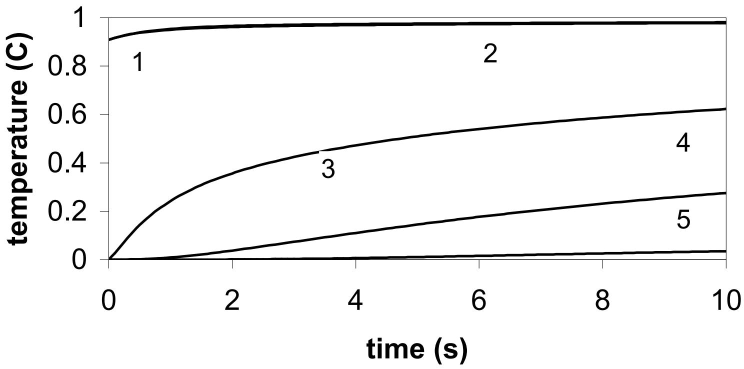

Figure 5 – Temperatures for the lineal solid

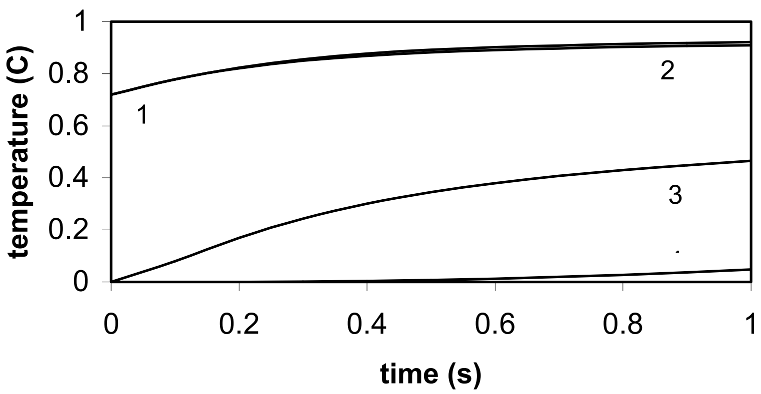

Figure 6 – Temperatures for the nonlineal solid

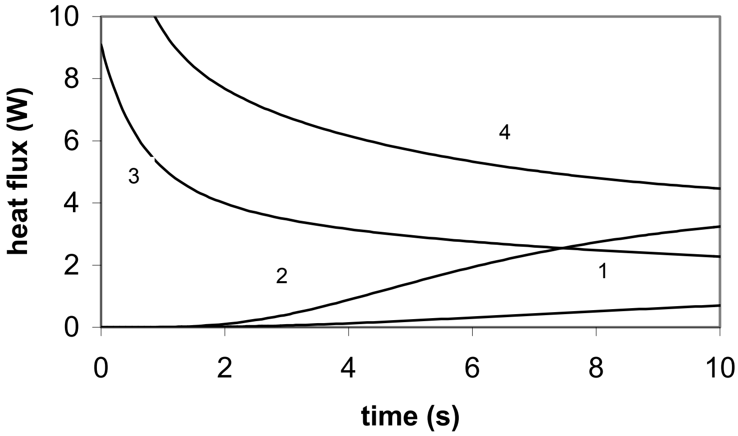

Figure 7 – Heat fluxes (lineal: curves 2 and 4, nonlineal: curves 1 and 3)

Second application. The left hand side of a square orthotropic flat plate (fig. 8) with temperature-dependent thermal conductivity is warmed by convection, while the right hand side is cooled by radiation. Both upper and lower sides are adiabatic.

Numerical values for the parameters are:

$L_x = L_y = 1$, $N_x = 20$, $N_y = 20$, $\Delta x = \Delta y = 0.05$,

$k_x = k_{x,0} + k_{x,1}$ (T-273) = 1 + 1(T-273), $k_y = k_{y,0} + k_{y,1}$ (T-273) = 0.1 + 0.1(T-273), ρ=1, $c_e = 400$,

$T_{ini} = 300$, $T_{ref,c} = 301$, h = 10, $T_{ref,r} = 300$, ε (emissivity)=1, σ (Boltzmann constant) = 5.67E-8

$j(x,0) = j(x,L_y) = j(0,y<L_y/4) = j(0, 3L_y/4<y<L_y) = 0$

$j(0, L_y/4<y<3L_y/4) = h_1 [T_{ref,c} - T(0,y)] = 10 [301 - T(0,y)]$

$j(L_x,y) = \varepsilon\sigma[(T(L_x,y)^4 - T_{ref,r}^4] = 5.67E-8 [(T(L_x,y)^4 - 300^4]$

With these values $R_{i\pm\Delta,j} = 0.025$, $R_{i,\, j\pm\Delta} = 0.25$, $C_{i,j} = 1$. Nonsteady temperatures and (convective and radiative) heat fluxes are shown in figures 9, 10 and 11. It is interesting to appreciate the crossing of temperature curves of the nonlineal case for locations $(L_x/2,L_y/2)$ y $(0,L_y)$, due to the great increase of the thermal conductivity k_y compared with k_x (due to the negligible value of $k_{x,0}$) as temperature increases. Time computing is less than 22 s in all cases.

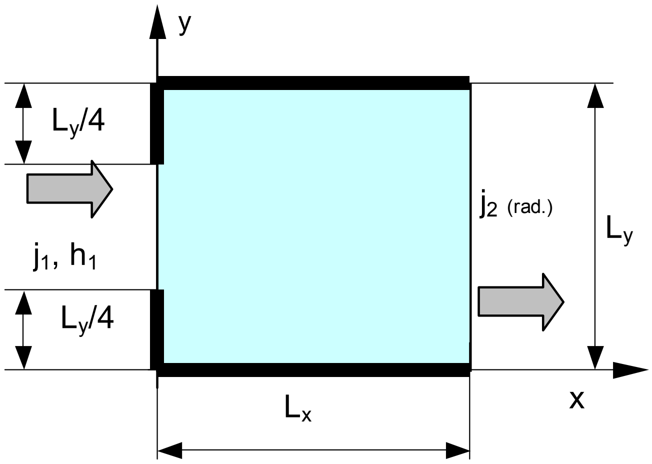

Figure 8 – Geometry of the second application

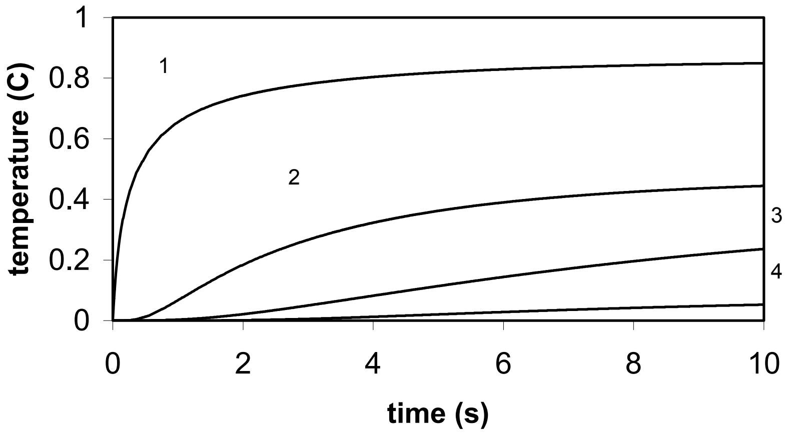

Figure 9 – Temperatures for the lineal solid

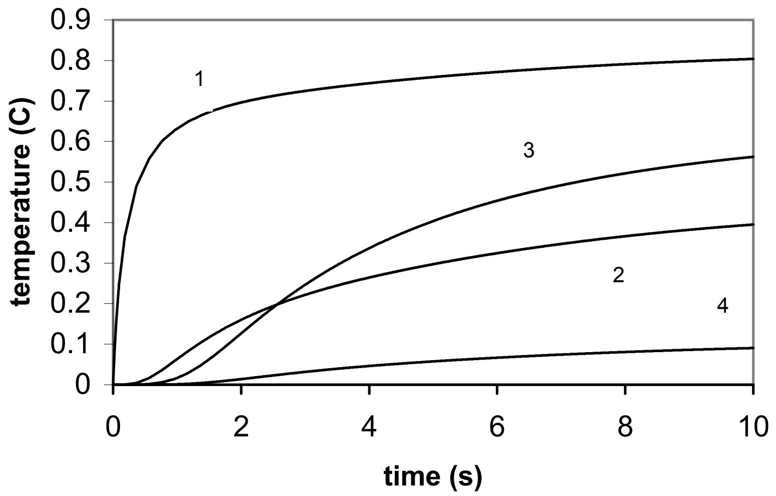

Figure 10 – Temperatures for the nonlineal solid

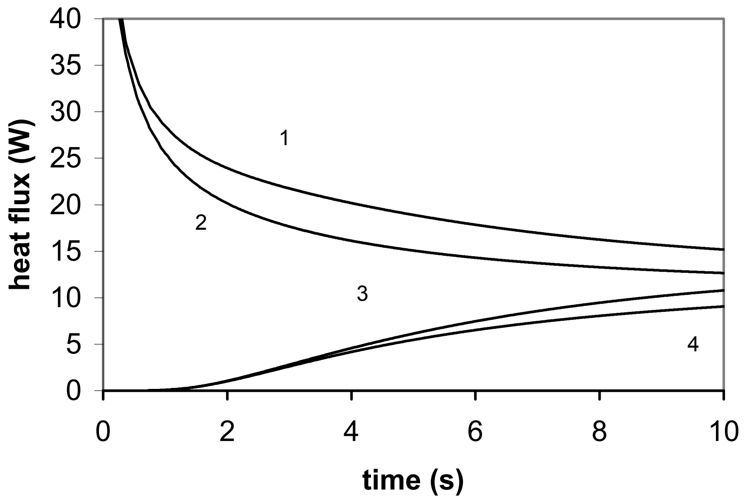

Figure 11 – Heat fluxes (lineal: curves 2 and 4, nonlineal: curves 1 and 3)

8 CONCLUSIONS

A Network model has been designed which permits the simulation and numerical solution of orthotropic heat conduction problems in a 2-D medium based on the network simulation method. The model can assumed arbitrary functional dependencies on temperature, both of thermal conductivity and specific heat. Arbitrary conditions (lineal or not) can be integrated in the same.

The network model, which only requires suitable circuit resolution software, has been applied to two specific problems with a number of volume elements of 100 and 400, the computing time in the least favourable case being 22 s.

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