Teaching compressible fluid dynamics with the help of a computer algebra system

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Abstract

This work shows how a computer algebra system (CAS) is applied in a course of advanced fluid mechanics. The authors put the focus on gas dynamics [1,4] and show how a CAS, in this case MAPLE 14 [3], can be used to teach this subject. It is known from the teaching experience of the authors that especially in gas dynamics some mathematical problems for the students appear. With the help of a CAS it is possible to shorten lengthy calculations for an instructor during lessons and students can work out problems on their own and improve their skills. An important fact are the visualization capabilities that a CAS offers. For example it is possible to study the behavior of equations or obtained solutions in a graphical manner or variables can be changed for a case study. In addition gas dynamics is a classical example for a multiphysics discipline in engineering. It is built up from the basic principles of continuum mechanics and bundles fluid mechanics, thermodynamics and chemistry. In this work the theoretical framework and two problems, concerning the de Laval nozzle are prepared with a CAS to present them in a modern course of compressible fluid dynamics.

Keywords: Computer Algebra System (CAS), fluid dynamics, compressible flow

1. Governing equations

The governing equations are obtained from the fundamental principles of continuum thermo-mechanics.

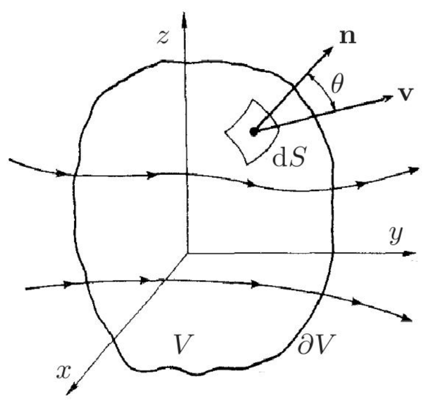

Figure 1: Fixed control volume for the derivation of the governing equations

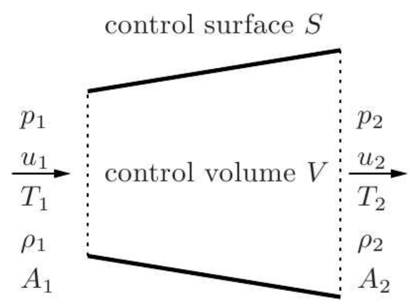

Figure 2: Finite control volume for quasi-one-dimensional flow

In detail these principles are:

Continuity equation: *"Mass can neither be created or destroyed"*

$$\frac{\partial}{\partial t}\int_V dV\rho = -\int_{\partial V} d\underline{S}\cdot\rho\underline{v} \tag{1}$$

Momentum equation: *"Time rate of change of momentum of a body equals the net force exert on it"*

$$\frac{\partial}{\partial t}\int_V dV\rho\underline{v} + \int_{\partial V} dS(\rho\underline{v}\cdot\underline{n})\underline{v} = \int_V dV\rho\underline{f} - \int_{\partial V} d\underline{S}\cdot p \tag{2}$$

Energy equation: *"Energy can neither be created nor destroyed; it can only change in form"*

$$\frac{\partial}{\partial t}\int_V dV\rho\left(e+\frac{v^2}{2}\right) + \int_{\partial V} d\underline{S}\cdot v\rho\left(e+\frac{v^2}{2}\right) = \int_V dV\rho\underline{\dot{q}} + \int_{\partial V} d\underline{S}\cdot p\underline{v} + \int_V dV\rho\left(\underline{f}\cdot\underline{v}\right) \tag{3}$$

Additionally to these equations, the constitutive behavior of the medium must be described. This is done by an equation of state connecting the thermodynamic variable pressure *p*, density *ρ* and temperature *T*. The simplest vaporous medium is the calorically perfect gas, described by the following equation of state:

$$pv = RT \tag{4}$$

In Eq.4 the specific volume *v=1/ρ* and the specific gas constant *R* is introduced. The equation of state describes a surface in a threedimenional *p,v,T* – space, as shown in Fig.3. A thermodynamical process can take place between two points on this surface. Additionally a thermodynamic relation for the internal energy *e* in the form

$$e = e(T,\rho) \tag{5}$$

is necessary. For a calorically perfect gas this relation is very simple and its derivation can be found in textbooks on thermodynamics, e.g. [2]. The relation is given by:

$$e = c_v T \tag{6}$$

where c_v is the specific heat at constant volume of the gas. In this work a quasi one-dimensional flow is investigated, i.e., *p=p(x), ρ=ρ(x), T=T(x), u=u(x)* and *A=A(x)* for the area of the stream tube, which means that only one spacial variable exists. Application of the governing equations to a one-dimensional steady adiabatic flow without body forces through the control volume, shown in Fig.2, gives

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \qquad \text{continuity equation} \tag{7}$$

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} dAp = p_2 A_2 + \rho_2 u_2^2 A_2 \qquad \text{momentum equation} \tag{8}$$

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2} \qquad \text{energy equation} \tag{9}$$

It is advantageous to introduce the enthalpy $h = e + \frac{p}{\rho}$ in the energy equation resulting in

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{10}$$

1.1 Isentropic process

From the fundamental energy principle the first law of thermodynamics follows. For a closed system it states that an increment change of internal energy caused by an increment change of work plus the increment change of other energies respectively heat. In the language of mathematics this is given by

$$de = \delta p + \delta w \tag{11}$$

where *d* denotes an exact differential and δ denotes a differential which is not exact. There exist different possibilities to supply heat to a thermodynamical system. For example, if heat is added reversible to the system then the increment in work is $\delta w = -pdv$ (see [2]), and the first law becomes

$$de = \delta q - pdv \tag{12}$$

In the further derivation it is necessary to introduce the second law of thermodynamics and additionally the state variable entropy. To understand the content of the second law, consider the following experiment. Consider an ice cube in a cup of hot tea. From experience it is clear that the ice will warm up and melt and the tea cool down. However, as long as energy is conserved the first law allows that the ice cube get cooler and the tea heats up, but this is not observed up today by anybody in reality. So it is obvious that nature imposes a condition in which direction a process will take place. This condition is the second law of thermodynamics. In mathematical form of the second law reads

$$ds \geq \frac{\delta q}{T} \tag{13}$$

If the process is reversible then in Eq.13 the equal sign is valid. For a calorically perfect gas and a reversible process the change in entropy is

$$\Delta s = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \tag{14}$$

The process is called isentropic if the entropy is constant, i.e. $\Delta s = 0$. From the last equation, the equation of state and the isentropic exponent

$$\kappa = \frac{c_p}{c_v} \tag{15}$$

The following very useful relations can be derived. They connect the state variables *p, T, v, ρ*. In detail they are

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-\frac{1}{\kappa-1}}, \quad \frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\kappa-1}} \quad \text{and} \quad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\kappa} = \left(\frac{T_2}{T_1}\right)^{-\frac{1}{\kappa-1}} \tag{16}$$

The last relation is the isentropic relation

$$pv^{\kappa} = const. \tag{17}$$

In Fig.4 this equation is plotted for a common value of the isentropic exponent $\kappa = 7/5$. At the end of this section it should be noted that an isentropic process is a process which is reversible and adiabatic.

1.2 The speed of sound

By definition, sound is a mechanical wave or disturbance propagating through a medium (gas, liquid or solid). The amplitude of the wave is small and the change of all thermodynamic properties across the wave front is also small. The speed at such disturbances propagate

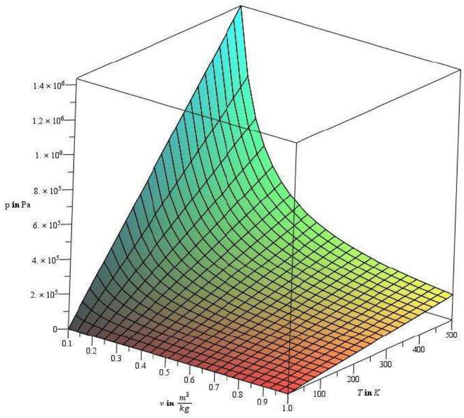

Figure 3: Surface defined by the equation State pv=RT for calorically a perfect gas

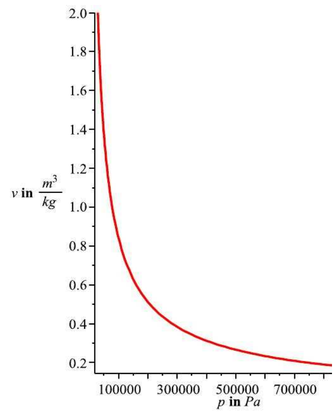

Figure 4: Isentropic process $pv^{\kappa} = 77472$ for $\kappa = 7/5$

through a medium is the speed of sound. It is one of the most important quantities in studying compressible flows. To calculate the velocity of sound the following gedankenexperiment is done. A sound wave propagates with the velocity *a*. If an observer moves with the wave, from his point of view the wave front is stationary and he will see the picture shown in Fig. 5.

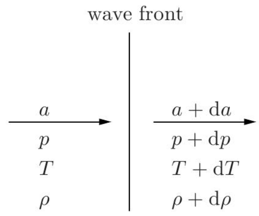

Figure 5: Schematic of a plane sound wave; state 1 upstream, state 2 downstream, speed of sound *a*, pressure *p*, density *ρ*, temperature *T* and their increments denoted by *da, dp, dρ, dT.*

Applying the equations of continuity, Eq.7, gives

$$\rho a = (\rho + d\rho)(a + da), \ \Rightarrow a = -\rho \frac{da}{d\rho} \qquad (18.1, 18.2)$$

The momentum, Eq.8, reads as:

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2, \Rightarrow da = -\frac{dp + a^2 d\rho}{2\rho a} \qquad (19.1, 19.2)$$

Substitution of the Eq.18.2 in Eq.19.2 gives

$$a^2 = \frac{dp}{d\rho} \qquad (20)$$

In the sense of thermodynamics crossing the front of the sound wave is process. By definition the changes of the field variables *T, p, ρ* across the front are slight, i.e. all gradients are small. All irreversible effects due to thermal conduction and friction are negligible and there is no heat addition to the fluid hence the process is adiabatic and reversible, i.e., crossing the wave front is an isentropic process (*s*=const.).

In a more precisely fashion the speed of sound is:

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s \qquad (21)$$

There exists a connection with the isentropic compressibility τ_S of the medium and the speed of sound given by:

$$\tau_s = -\frac{1}{v}\left.\frac{\partial v}{\partial p}\right|_s, \Rightarrow a = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\frac{v}{\tau_s}} \qquad (22)$$

This relation gives an explanation for the high speed of sound in materials with a low compressibility e.g. fluids at normal conditions or solid materials. In Tab.1 the speed of sound for some materials is given.

Table 1: Speed of sound for different materials [5]

Medium	Air		Hydrogen	Water	Copper	Aluminium	Iron
T in °C	0	100	0	25	25	25	25
a in m/s	331	386	1290	1490	3560	5100	5130

For an isentropic process (pv^κ =const.) of a calorically perfect gas ($pv=RT$) the speed of sound reads as:

$$a = \sqrt{\kappa R T} \qquad (23)$$

This is an important equation in mechanics of compressible fluids. It shows directly the dependence of a on temperature T and the sort of gas through R and κ. To connect the velocity of a medium to the speed of sound at a certain point in the flow field, the Mach number M is introduced. It is defined as

$$M = \frac{u}{a} \qquad (24)$$

In words this relation states: "*The Mach number is the quotient of the local fluid velocity and the local speed of sound*". From its definition it is clear that M is also a local quantity. With the Mach number it is possible to categorize compressible flows into several classes. For $M = 1$ the flow is sonic, if $M < 1$ then it is subsonic and if $M > 1$ the flow is supersonic. In aero- and astronautics there exists situations in which $M >> 1$, then the flow is called hypersonic.

1.3 Concept of total behavior

It is advantageous to define two hypothetical fluid states. The first one is the so called total or stagnation state. It is defined in the following way. Consider a fluid element at the state p, T, ρ, u. The element is now slowed down isentropically to zero velocity. At this state the element have the stagnation or total pressure p_t, the stagnation or total temperature T_t and the stagnation or total density ρ_t. From the energy equation Eq.9 and the enthalpy h it follows that

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_t \qquad (25)$$

The total temperature can be written in terms of the temperature and the Mach number and the isentropic exponent as:

$$\frac{T_t}{T} = 1 + \frac{\kappa - 1}{2} M^2 \qquad (26)$$

Inserting the isentropic relation $pv^{\kappa}=const.$, the expression for the pressure follows as:

$$\frac{p}{p_t}=\left(\frac{\rho_t}{\rho}\right)^{\kappa}, \quad \frac{p}{p_t}=\left(\frac{T_t}{T}\right)^{\frac{\kappa}{1-\kappa}} \tag{27.1, 27.2}$$

Substitution of Eq.26 in Eq.27.1 and Eq.27.2 allows to write expressions for the total pressure and for the total density in terms of *ρ, κ, M* as

$$\frac{p}{p_t}=\left[1+\frac{\kappa-1}{2}M^2\right]^{\frac{\kappa}{\kappa-1}} \quad , \quad \frac{\rho_t}{\rho}=\left[1+\frac{\kappa-1}{2}M^2\right]^{\frac{1}{\kappa-1}} \tag{28}$$

The second useful hypothetical state of a fluid element is the so called *-state, reached in the following way. Consider again a fluid element at an arbitrary state given by *p, T, ρ, M*. Now imagine to speed up (if $M < 1$) or slow down (if $M > 1$) the fluid element until the Mach number $M = 1$ is reached. In the new state the temperature is T^*. The speed of sound at this hypothetical state is

$$a^* = \sqrt{\kappa R T^*} \tag{29}$$

2. Isentropic Flow of a calorically perfect gas through variable-area ducts

Writing the equation of continuity for the control volume in Fig.6 gives:

$$\rho^* u^* A^* = \rho u A \quad \Leftarrow u^* = a^* \tag{30}$$

$$\frac{A}{A^*}=\frac{\rho^*}{\rho_t}\frac{\rho_t}{\rho}\frac{a^*}{u} \tag{31}$$

Introducing total quantities and specialization for sonic conditions the following equation is obtained:

$$\left(\frac{A}{A^*}\right)^2=\frac{1}{M^2}\left[\frac{2}{\kappa+1}\left(1+\frac{\kappa-1}{2}M^2\right)\right] \tag{32}$$

The Eq.32 is one of the most important relations in gas dynamics. Therefore it is instructive to visualize it. In Fig.7 the Mach number M is drawn as a function of the area quotient $A_v=A/A^*$ with the isentropic exponent κ as parameter.

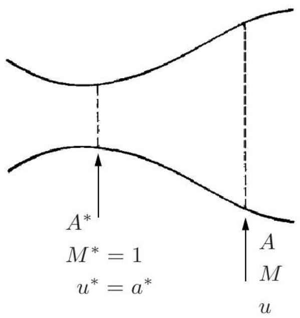

Figure 6: Geometry for the derivation of the area-mach number relation

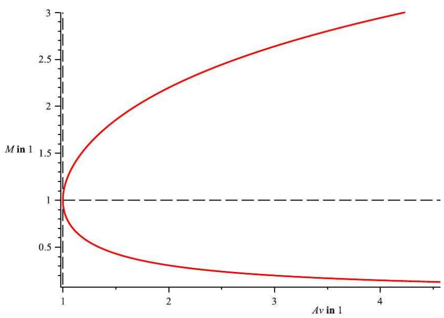

Figure 7: Area-mach number relation with isentropic exponent κ as parameter

From this picture it can be seen that how a duct must be designed to accelerate a flow from the subsonic regime into the sonic or supersonic regime. Starting from rest (gas in a pressurized vessel) the area of the duct must decrease. At the point $A_v=1$ the cross section area of the duct is exact the critical area and the velocity of the gas is the speed of sound. If it is necessary to increase the gas velocity further, an outstanding effect occurs. Contrary to expectations, the area cross section of the duct must increase from this point on. Such a device is called de Laval nozzle after the Swedish engineer Gustaf de Laval, who developed it in 1880's for the usage in a steam turbine, which was also invented by him [6].

3. Examples

The following two examples show how CAS can be used to solve problems in compressible fluid dynamics and how it can be utilized to visualize results. The authors have chosen MAPLE 14 [3] for the computations in this work. All worksheets can be obtained from the authors.
The geometry of the nozzle is given from the construction by the ratio A_0/A as function of the normalized nozzle length x/L as:

$$\frac{A_0}{A}(x/L) = K_0 + K_1\frac{x}{L} + K_2\left(\frac{x}{L}+c_2\right)^2 + K_3\left(\frac{x}{L}+c_2\right)^3 + K_4\left(\frac{x}{L}+c_2\right)^4 \qquad (33)$$

This area ratio can be visualized in a simple manner by the CAS, it is drawn in Fig.8.

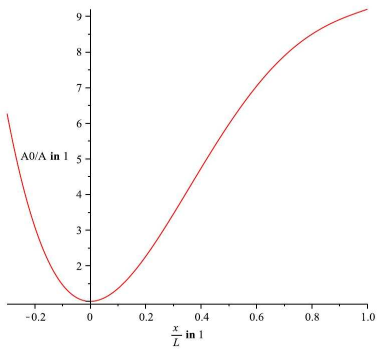

Figure 8: Area ratio $A_0/A(x/L)$ of the de Laval Nozzle investigated in example 1 and 2

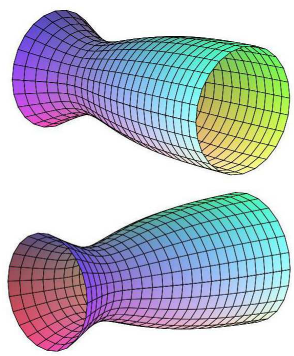

Figure 9: Shape of the de Laval nozzle investigated in example 1 and 2.

The nozzle should have a circular cross section. Therefore the function area ratio in Eq.33 can be resolved in terms of the nozzle diameter ratio. It is instructive to show the geometry of the nozzle. This is also a simple task if a CAS is used. For example a ratio of d_0/L = 5 gives the shape shown in Fig.9.

3.1. Example 1, M_0=0,9

The function A_0/A and the Mach number M_0 of a convergent-divergent nozzle is given. A_0 is the through of the nozzle and M_0 is the corresponding Mach number. The gas flowing through the nozzle has in isentropic exponent κ. A one-dimensional steady state isentropic flow is assumed. The specific flow variables p/p_t, T/T_t, ρ/ρ_t and the Mach number M as a function of the relative nozzle length x/L are sought.
Following data are given: κ=1.4, M_0=0.9, T_t=400K, p_t=10bar, R=287J/kgK, ρ_t=8,7108kg/m³. The result for $M(x/L)$ is given in Fig.10 and the specific flow quantities are plotted in Fig.11.

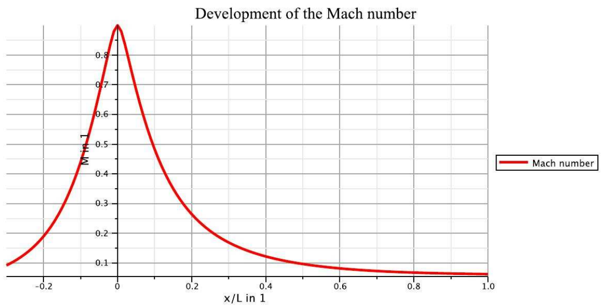

Figure 10: Mach number *M(x/L)* for $M_0=0.9$

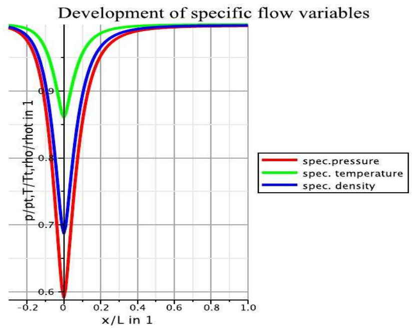

Figure 11: Specific flow variables for $M_0=0.9$

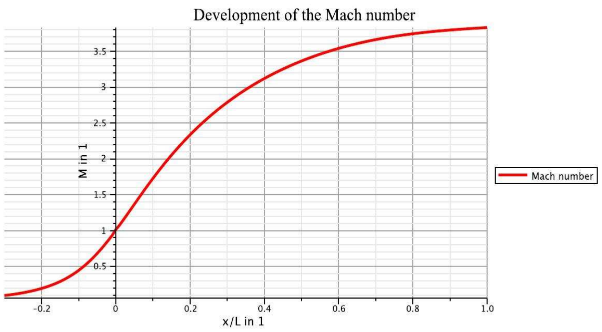

Figure 12: Mach number *M(x/L)* for $M_0=1$

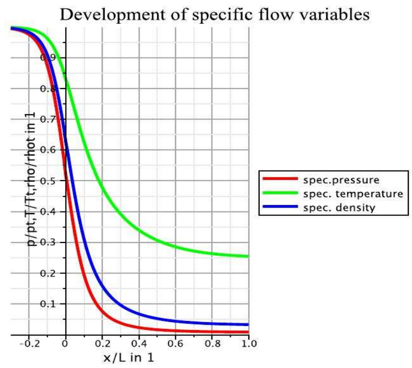

Figure 13: Specific flow variables for $M_0=1$

3.2. Example 2, M_0=1

Expansion to M>1, sonic conditions in the nozzle through : The function A_0/A is the same as in Example 1. The Mach number M_0 is now $M_0=M^*=1$. All other flow data are the same as in Example 1. The result for M(x/L) is given in Fig. 12 and the specific flow quantities are plotted in Fig.13.

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