

Valuable mathematical tools in engineering education

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Abstract – This work is an illustrated application of tensor calculus in conjunction with a CAS (computer algebra system) in our case MAPLETM. As an example we treat a problem pertaining to the field of thermoelasticity, which in turn belongs to the larger class of continuum mechanics problems. As a first step of our deductive approach the governing equations are derived and after introducing an adequate metric the problem is made accessible to a solution procedure. In this step we use the CAS to build the terms of the resulting Lamé-Navier equation in appropriate coordinates. We avoid the use of built-in functions in the CAS and show how the terms can be assembled. The solution of the resulting differential equation is carried out in two ways. First an analytical solution is obtained by means of the CAS. Second a numerical solution is computed by using the FEM (finite element method). This method is today a standard technique for solving partial differential equations (PDE) in physics and engineering. It should be pointed out that this example problem is also calculable by hand. However, in more complicated cases CASs or the FEM are the only possibility to solve such a problem. Today CASs are useful tools for physicists and engineers in everyday life. Therefore, the introduction of such tools in the education of engineers should be promoted.

Index Terms – computer algebra system, continuum mechanics, differential equations, numerical methods, tensor calculus.

Introduction

When we try to represent the nature by physical laws we introduce an adequate coordinate system. In this coordinate system we set up equations which should describe the process. The most fundamental postulate is that the content of a physical law is the same regardless of which reference frame is chosen for the description. It is the chief aim of *tensor calculus* to investigate the transformation behavior of physical quantities when we change from one coordinate system to another. The originators of tensor calculus are Ricci and Levi-Civita [1]. With their work they defined the tensor calculus as systematic branch of mathematics in the late 19th century. Before that the mathematicians Gauss, Riemann and Christoffel introduced the concept of tensors in the development of differential geometry. One of the first extensive and most powerful application of Ricci's calculus (e.g. tensor calculus) was Einstein's general theory of relativity [2].

With this work the advantage of formulating physical equations especially in continuum mechanics with the tensor formalism will be shown. If we can express an equation in tensor form we can write it in every other coordinate system. The extensive calculations which are arising are done with a CAS (computer algebra system), in our case MAPLETM. With this approach it is possible to concentrate on concepts and not on time consuming number crunching. The notation used in this work is the same as in [3] and [4].

Theoretical foundations

In this section the basic equations of linear thermoelasticity are developed and formulated in tensor notation. A good overview about thermoelasticity is given in [5] or in [6], but they use only vector notation or Cartesian tensor notation. General tensor notation is used in [4] and [7]. First the kinematic relations are formulated in curvilinear coordinates. The equation of motion is shown and specialized for the static case. Particular attention is paid to constitutive equations. The constitutive equation for linear elasticity e.g. *Hooke's law* is given in general form. Then it is expanded to thermoelasticity by application of the *Duhamel-Neumann relation*. After presenting the theoretical basis a special application is shown. A thermomechanically loaded cylindrical specimen is investigated and the solution of this problem is given in an analytical and a numerical way. The analytical computations could be done manually or in this work by a CAS, here MAPLETM V10. An overview about the functionality is given in [8]. For the numerical solution the FEM (finite element method) is used, which is a standard method to solve partial differential equations. In this work the finite element system ABAQUS V6.6 [9] is used.

The location of a point P in a three-dimensional space is described by the curvilinear coordinates Θ^k and the time t . The motion of a body is defined by the mapping

$$x^i = x^i(X^j(\Theta^k), t) \quad \text{and by the inverse mapping} \quad X^i = X^i(x^j(\Theta^k), t) \quad \text{with} \quad i, j, k = 1 \dots 3. \quad (1)$$

The first one is called material or *Lagrange* description and the second one spatial or *Eulerian* description of the motion. The differentials of the X^i and x^i are given by

$$dX^j = \frac{\partial X^j}{\partial \Theta^i} d\Theta^i = X^j_{,i} d\Theta^i = \mathbf{G}_i d\Theta^i, \quad \text{and} \quad dx^j = \frac{\partial x^j}{\partial \Theta^i} d\Theta^i = x^j_{,i} d\Theta^i = \mathbf{g}_i d\Theta^i, \quad (2)$$

where \mathbf{G}_i and \mathbf{g}_i are the covariant bases. In this work the *Einstein notation* is applied, for example $\psi = x_i y^j$ means

$\psi = \sum_{i=1}^n x_i y^1 + x_2 y^2 \dots x_n y^n$. For the partial derivative of a quantity with respect to the coordinates the notation $(\dots)_{,k}$ is used.

Uppercase characters refer to the reference configuration, whereas lowercase characters denote the current configuration. The square of the infinitesimal line element in the undeformed reference configuration is

$$dS^2 = G_{ij} dX^i dX^j, \quad \text{and in the deformed configuration} \quad ds^2 = g_{ij} dx^i dx^j. \quad (3)$$

In the former equation the covariant metric coefficients $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$, $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ are introduced. With the difference of the squared infinitesimal line elements it is possible to define a variable which can be used to describe the deformation of the body

$$ds^2 - dS^2 = g_{ij}dx^i dx^j - G_{ij}dX^i dX^j \quad \text{respectively} \quad ds^2 - dS^2 = 2E_{ij}d\Theta^i d\Theta^j. \quad (4)$$

The quantity which is used to describe the deformation in the reference configuration is the *Green-Lagrange* deformation tensor. It is defined by

$$\mathbf{E} = E_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \quad \text{and the components of this tensor are} \quad E_{ij} = \frac{1}{2}(g_{ij} - G_{ij}). \quad (5)$$

Now the physical meaning of the Green-Lagrange deformation tensor is clear. It is the difference between the metric of the current configuration and the reference configuration. Therefore E_{ij} is a suitable quantity to describe the deformation of a material body. If there is no difference in the metric, i.e., $g_{ij} = G_{ij}$ and $E_{ij} = 0$ then the body is called *rigid*. In a deformed body the location of material point is described by

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \quad \text{or in components} \quad x^i(\Theta^k, t) = X^i(\Theta^k, t) + u^i(\Theta^k, t) \quad (6)$$

The quantity u^i is the *displacement vector*. From (6) the curvilinear basis follows by differentiating with respect to the coordinates Θ^k .

$$\mathbf{g}_i = \mathbf{G}_i + \mathbf{u}_{,i}. \quad (7)$$

For the coordinates of the metric tensor

$$\mathbf{g}_i \cdot \mathbf{g}_j = g_{ij}, \quad g_{ij} = G_{ij} + \mathbf{G}_i \cdot \mathbf{u}_{,j} + \mathbf{G}_j \cdot \mathbf{u}_{,i} + \mathbf{u}_{,j} \cdot \mathbf{u}_{,i} \quad (8)$$

The displacement vector and its derivative is

$$\mathbf{u} = u_k \mathbf{g}^k, \quad \mathbf{u}_{,j} = u_{k|j} \mathbf{g}^k \quad (9)$$

where the notation $(\dots)_{|k}$ is introduced for the *covariant derivative*. This result can be substituted in (5) for the Green-Lagrange tensor

$$E_{ij} = \frac{1}{2}(g_{ij} - G_{ij}), \quad E_{ij} = \frac{1}{2}(\delta_i^k u_{k|j} + \delta_j^k u_{k|i} + g^{kl} u_{k|i} u_{l|j}), \quad E_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i} + u_{k|i} u^k{}_{|j}) \quad (10)$$

where δ_j^k is the Kronecker delta. In the *linearized theory of elasticity* the quadratic terms in (10) are neglected and there is no disagreement between the deformed and the undeformed configuration. From the Green-Lagrange tensor the *linearized deformation tensor* follows as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}) \quad (11)$$

The second part is the derivation of the *equation of motion*. From Newton's second law for a continuum we get

$$\int_S dS t^i + \int_V dV \rho b^i = \partial_t \int_V dV \rho v^i \quad (12)$$

with the density ρ the body force b^i and the velocity v^i . This is the *global form* or *integral form* of the equation of motion. The stress vector t^i can be expressed in terms of the *Cauchy stress tensor* σ^{ij} by $t^i = \sigma^{ij} n_j$. This relation is called *Cauchy's lemma*. It can be shown that the stress tensor σ^{ij} is symmetric ($\sigma^{ij} = \sigma^{ji}$). This is direct result from the postulate of conservation of angular momentum. (Cauchy's second equation of motion, deduced by L. Euler). With the *divergence theorem of Gauss* it is possible to rewrite the equation of motion. The surface integral can be expressed by a volume integral and the stress vector by the stress tensor.

$$\int_V dV (\sigma^{ji}{}_{|i} + \rho b^j - \rho \partial_t v^j) = 0. \quad (13)$$

Equation (13) must be true for each volume. Therefore we get with $v = \dot{u}$ the *local form* or *differential form* of the equation of motion as

$$\sigma^{ij}{}_{|j} + f^i = \rho \ddot{u}^i. \quad (14)$$

The final necessary part is a relation between the stresses and strains. Such a relation is called *constitutive equation*. The development of constitutive equations for different kinds of materials is one of the major tasks in materials science and computational mechanics. In the field of linearized elasticity the constitutive equation is called *Hooke's law* and it is given by

$$\sigma^{ij} = E^{ijkl} \varepsilon_{kl}. \quad (15)$$

The tensor E^{ijkl} is of fourth order and is called *material* or *elasticity tensor*. Due to the symmetry condition of the Cauchy stress tensor, the number of components in the elasticity tensor is reduced from 81 to 21. If we deal with an *isotropic* material then only two constants are necessary to describe the material behavior. These constants are the *Young's modulus* E (modulus of elasticity) and the *Poisson ratio* ν . For further derivations it is useful to introduce the *shear modulus* G

$$G = \frac{E}{2(1+\nu)}. \quad (16)$$

The complete set of field equations in the theory of linear elasticity are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}), \quad \sigma^{ij}{}_{|j} + f^i = \rho \ddot{u}^i, \quad \sigma^{ij} = G \left(g^{ik} g^{jl} + g^{il} g^{jk} + \frac{2\nu}{1-2\nu} g^{ij} g^{kl} \right) \varepsilon_{kl}, \quad \varepsilon_{ij|kl} e^{ikm} e^{jln} = 0. \quad (17)$$

These equations are called *kinematic relation* (17.1), *equation of motion* (17.2), *constitutive law* (17.3) and the *compatibility conditions* (17.4). The symbol e^{ikm} in the last equation is the Levi-Civita symbol, also called the permutation symbol. e is

1 if (i, k, m) is an even permutation of $(1, 2, 3)$, -1 if it is an odd permutation, and 0 if any index is repeated. The number of variables in this set of 15 partial differential equations is also 15 (6 components of the symmetric strain tensor ε_{ij} , 6 components of the symmetric stress tensor σ^{ij} and 3 components of displacement vector u^i). With this set of equations it is possible to describe small motion and equilibrium of an elastic body. In the case of elastostatics the equation of motion reduces to

$$\sigma^{ij}|_j + f^i = 0. \quad (18)$$

In this paper we are interested in static problems of thermoelasticity. To investigate such problems we have to extend the constitutive law from linear elasticity to linear thermoelasticity. In the field of thermoelasticity the constitutive relation (17.3) has to be extended to take into account the thermal expansion effects. The resulting relation is

$$\sigma^{ij} = E^{ijkl}(\varepsilon_{kl} - \alpha^{ij}\theta). \quad (19)$$

This equation is the inverse form of the so called *Duhamel-Neumann relation*. In this relation the second-rank tensor α^{ij} is called *thermoelasticity tensor*. In the linearized theory of thermoelasticity we assume that E^{ijkl} is not a function of temperature T and that α^{ij} is not a function of ε_{kl} [10]. The number of independent entries in α^{ij} depends on the degree of anisotropy of the material. For example if we deal with an orthotropic material we need only the components α^{ij} in the principal directions. The simplest material is the *isotropic* one, for which the derivation of the Duhamel-Neumann relation is shown. The main idea is that the total strain ε_{tot} can be decomposed in a mechanical strain $\varepsilon_{\text{mech}}$ and a thermal strain $\varepsilon_{\text{therm}}$.

$$\varepsilon_{j(\text{tot})}^i = \varepsilon_{j(\text{mech})}^i + \varepsilon_{j(\text{therm})}^i, \quad \text{where} \quad \varepsilon_{j(\text{therm})}^i = \alpha_T \theta \delta_j^i \quad \text{with} \quad \theta = T - T_0. \quad (20)$$

It is common to use absolute temperatures and to set $T_0 = 0$, and $\theta = T$. In order to take the thermal strains into account we have to reformulate the constitutive equation which gives after some manipulations

$$\sigma_s^i = \frac{E}{1-\nu} \left(\varepsilon_s^i + \frac{\nu}{1-2\nu} \delta_s^i \varepsilon_k^k \right) \quad \text{and from this we get with} \quad \delta_i^i = 3, \quad \sigma_i^i = \frac{E}{1-2\nu} \varepsilon_i^i. \quad (21)$$

With this result it is possible to compute ε_j^i . Adding the thermal strain gives the final expression often called *inverse Hooke's law* for ε_j^i in terms of the mixed stress tensor σ_j^i and the elastic constants E and ν

$$\varepsilon_j^i = \frac{1}{2G} \left(\sigma_j^i - \frac{1}{1+\nu} \sigma_m^m \delta_j^i \right) + \alpha_T T \delta_j^i. \quad (22)$$

This equation is the Duhamel-Neumann relation for a mechanically and thermally homogeneous isotropic body. Inversion of (22) gives

$$\sigma^{ij} = \frac{E}{1+\nu} \left(\varepsilon^{ij} + \frac{\nu}{1-2\nu} \varepsilon_m^m g^{ij} - \frac{1+\nu}{1-2\nu} \alpha_T T g^{ij} \right). \quad (23)$$

It is possible to reduce the set of field equations (17.1) – (17.3) to three partial differential equations for the displacement field \mathbf{u} and the temperature T . This is done by building the covariant derivative of the constitutive law and inserting it together with the kinematic relation into the equation of motion. The resulting set of equations is called *Lamé-Navier equations*

$$\mu u^i|_j + (\lambda + \mu) u^j|_j - (3\lambda + 2\mu) \alpha_T T|_i = 0 \quad (24)$$

The parameters λ, μ are called *Lamé-Navier parameters*. Some useful relations between these parameters are

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} = G, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \quad (25)$$

Now we can start to solve the problem. The problem is formulated in cylindrical coordinates, where the location of a point P with respect to a Cartesian basis \mathbf{e}_i is given by

$$\mathbf{r} = r \cos \varphi \mathbf{e}_1 + r \sin \varphi \mathbf{e}_2 + z \mathbf{e}_3 \quad (26)$$

This is shown in figure 1 where also the coordinate surfaces are shown. The covariant basis \mathbf{g}_i is defined by

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \Theta^i} = \partial_{\Theta^i} \mathbf{r} \quad \text{so the basis vectors are:} \quad \mathbf{g}_1 = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2, \quad \mathbf{g}_2 = -r \sin \varphi \mathbf{e}_1 + r \cos \varphi \mathbf{e}_2, \quad \text{and} \quad \mathbf{g}_3 = \mathbf{e}_3. \quad (27)$$

The covariant and contravariant metric coefficients are defined by

$$\mathbf{g} = g^{ij} \mathbf{g}_i \mathbf{g}_j \quad \text{and} \quad \mathbf{g} = g_{ij} \mathbf{g}^i \mathbf{g}^j. \quad (28)$$

In cylindrical coordinates the nonvanishing metric coefficients are

$$g_{11} = g_{33} = 1, \quad g_{22} = r^2 \quad \text{and} \quad g^{11} = g^{33} = 1, \quad g^{22} = \frac{1}{r^2}. \quad (29)$$

With the metric defined by (28) it is possible to derive the *Christoffel symbols*. The symbols are defined by

$$\mathbf{g}_{l,m} = \Gamma_{lm}^k \mathbf{g}_k, \quad \text{and in terms of the metric coefficients} \quad \Gamma_{lm}^k = \frac{1}{2} g^{kn} (g_{mn,l} + g_{nl,m} - g_{lm,n}). \quad (30)$$

In cylindrical coordinates there exist only three nonvanishing Christoffel symbols. These are

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}. \quad (31)$$

In (31) the symmetry condition of the Christoffel symbols with respect to the lower indices is used. The Lamé-Navier equations in cylindrical coordinates can now be formulated. For the further solution procedure only the equation in r -direction

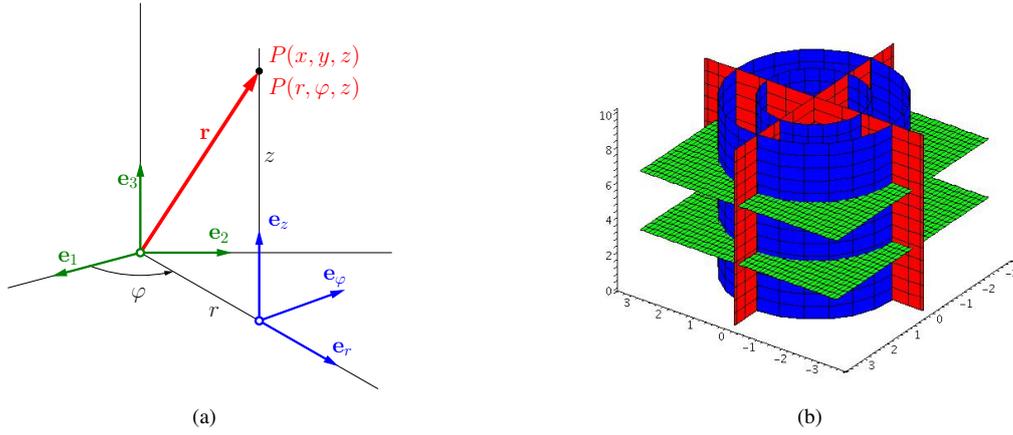


FIGURE 1

(a) Cylindrical coordinates r, φ, z with basis vectors $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z$ and Cartesian coordinates x, y, z with basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$; position vector \mathbf{r} of point $P(r, \varphi, z)$ or $P(x, y, z)$;
 (b) Coordinate Surfaces, $r = \text{const.}$ blue, $\varphi = \text{const.}$ red, $z = \text{const.}$ green

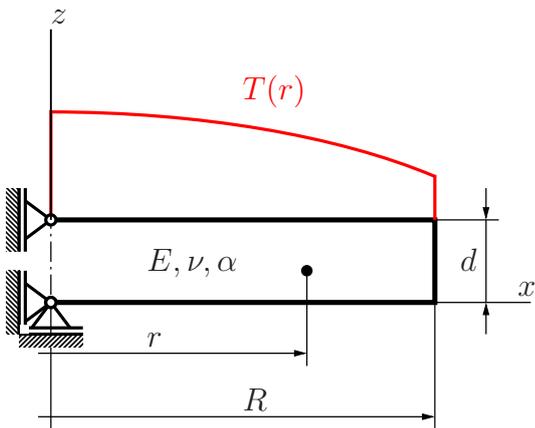
expressed in physical components will be considered here.

$$\begin{aligned}
 & (\lambda + \mu) \partial_r \left(\frac{1}{r} \partial_r r u_r + \frac{1}{r} \partial_\varphi u_\varphi + \partial_z u_z \right) + \mu \left(\partial_{rr} u_r + \frac{1}{r} \partial_r u_r + \frac{1}{r^2} \partial_\varphi \partial_\varphi u_r + \partial_{zz} u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \partial_\varphi u_\varphi \right) \\
 & - (3\lambda + 2\mu) \alpha \partial_r T r = 0.
 \end{aligned} \tag{32}$$

The detailed derivation of the three partial differential equations is given in the MAPLE worksheet in the Appendix.

Problem

In figure 2 a sketch of the analyzed problem is given. The stress-, strain- and displacement-fields in a circular disc or cylinder loaded by a radial temperature distribution are investigated. Dependent on the ratio of thickness d to radius R a plane stress or plane strain state is present in the body. Additionally the geometry and the properties of the isotropic linear elastic material (Youngs' modulus E , Poisson ratio ν and the coefficient of thermal expansion α) are given in this sketch. From the Lamé-Navier equations the differential equation for the radial displacement $u(r)$ is obtained. Due to symmetry



$\frac{d}{R} \ll 1 \dots$ plane stress
 $\frac{d}{R} \gg 1 \dots$ plane strain
 $\frac{d}{R} \approx 1 \dots$ 3-D-state

$$E = 2.1 \cdot 10^5 \text{ N/mm}^2$$

$$\nu = 0.3$$

$$\alpha = 12 \cdot 10^{-6} \text{ 1/K}$$

$$R = 11.25 \text{ mm}$$

$$d = 1 \text{ mm}$$

stress and strain free state at $T^* = 273.15 \text{ K}$

FIGURE 2

Circular disc or cylinder with radial temperature distribution; geometry boundary conditions and material properties.

conditions the partial differential equation (32) is transformed to an ordinary differential equation

$$(\lambda + 2\mu) d_r \left[\frac{1}{r} d_r (r u_r) \right] = (3\lambda + 2\mu) \alpha d_r T, \quad \text{or} \quad d_r \left[\frac{1}{r} d_r (r u_r) \right] = \alpha (1 + \nu) d_r T. \tag{33}$$

If the solution for the plane strain case is of interest, the variables E, ν, α have to be replaced by E_1, ν_1, α_1 . These variables are

$$E_1 = \frac{E}{1 - \nu^2}, \quad \nu_1 = \frac{\nu}{1 - \nu}, \quad \text{and} \quad \alpha_1 = \alpha (1 + \nu). \tag{34}$$

To solve the linear differential equation (33) for the displacement $u(r)$ it has to be integrated twice. The general solution of (33) for the radial displacement is given by

$$u(r) = \alpha \frac{(1+\nu)}{r} \int_0^r dr' T(r') r' + C_1 r^2 + C_2. \quad (35)$$

From the stress boundary conditions the constants of integration can be determined. The displacement field is connected with the stress field via Hooke's law and the strain displacement relations. The six components of the Cauchy stress tensor are reduced to three because we are dealing with a two-dimensional plane stress problem. These stresses are two normal components and one shear component. These are

$$\sigma_{rr} = -\frac{\alpha E}{r^2} \int_0^r dr' T(r') r' + \frac{C_1 E}{1-\nu} - \frac{C_2 E}{r^2(1+\nu)}, \quad \sigma_{\varphi\varphi} = \frac{\alpha E}{r^2} \int_0^r dr' T(r') r' - \alpha E T(r) + \frac{C_1 E}{1-\nu} + \frac{C_2 E}{r^2(1+\nu)}, \quad \sigma_{r\varphi} = 0. \quad (36)$$

Due to the fact that the cylindrical surface must be traction free, the stress σ_{rr} must satisfy the following conditions

$$\sigma_{rr}(a) = 0, \quad \sigma_{rr}(0) \neq \infty. \quad (37)$$

Inserting those in (36) yields the solution of the considered thermoelastic problem

$$\begin{aligned} \sigma_{rr} &= \alpha E \left[\frac{1}{R^2} \int_0^R dr T(r) r - \frac{1}{r^2} \int_0^r dr' T(r') r' \right], \quad \sigma_{\varphi\varphi} = \alpha E \left[\frac{1}{R^2} \int_0^R dr T(r) r + \frac{1}{r^2} \int_0^r dr' T(r') r' - T(r) \right], \\ u_r &= \frac{\alpha}{r} \left[(1-\nu) \left(\frac{r}{R} \right)^2 \int_0^R dr T(r) r + (1+\nu) \int_0^r dr' T(r') r' \right]. \end{aligned} \quad (38)$$

When the radius is zero the expression for the stresses and displacement becomes indetermined. If there is no singularity in the temperature distribution at $r = 0$ the integrals can be computed without a problem. The stresses at the position $r = 0$ can be determined by application of *l'Hôpital's* rule to the limits

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r dr' T(r') r' = 0, \quad \lim_{r \rightarrow 0} \frac{1}{r^2} \int_0^r dr' T(r') r' = \frac{1}{2} T(0) \text{ so that } \sigma_{rr}(0) = \sigma_{\varphi\varphi}(0) = \alpha E \left[\frac{1}{R^2} \int_0^R dr T(r) r - \frac{1}{2} T(0) \right]. \quad (39)$$

The integrals in the general solution (38) can be evaluated straightforward by hand or by the help of a CAS. If a CAS is applied for this task the field variables are obtained in a very quick way. For example an instructor could show this in his lecture online. So it is easy for students to get an impression how different loading conditions influence the solution.

Analytical solution

The following boundary conditions were investigated: (1) T is constant, (2) T is a linear function of radius r and, (3) T is a Gaussian function dependent on radius r .

Case 1, $T(r) = \text{const.}$

The temperature is

$$T(r) = T_0, \quad T_0 = 723.15 \text{ K} \quad (40)$$

and therefore the integrals in (38) can easily be evaluated. For the stresses and the displacement the following expressions are achieved

$$\sigma_{rr} = 0, \quad \sigma_{\varphi\varphi} = 0, \quad u_r = r \alpha T_0. \quad (41)$$

In figure 3.1 the radial stress σ_{rr} , circumferential stress $\sigma_{\varphi\varphi}$ and the radial displacement u_r are plotted as a function of radius r .

Case 2, $T(r) = f(r)$

In this case the temperature is a linear function of the radius. It is given by

$$T(r) = T_0 - \frac{T_0 - T_1}{R} r \quad T_0 = 723.15 \text{ K} \quad \text{and} \quad T_1 = 393.15 \text{ K}. \quad (42)$$

Substitution of this function and solving the integrals in (38) results in

$$\sigma_{rr} = -\frac{\alpha E}{3R} (T_0 - T_1) (R - r), \quad \sigma_{\varphi\varphi} = -\frac{\alpha E}{3R} (T_0 - T_1) (R - 2r) \quad \text{and} \quad (43)$$

$$u_r = \frac{\alpha r}{3R} \left\{ T_1 [r + R + \nu(r - R)] + T_0 [2R - r + \nu(R - r)] \right\}. \quad (44)$$

In figure 3.2 the radial stress σ_{rr} , circumferential stress $\sigma_{\varphi\varphi}$ and the radial displacement u_r are plotted as a function of radius r .

Case 3, $T(r) = f(r)$

The temperature is now a Gaussian function of the radius. This distribution is described by

$$T(r) = T_0 \frac{1}{\sqrt{2\pi}} e^{-cr^2} + T_1 \quad \text{with} \quad T_1 = 448.15 \text{ K}, \quad T_0 = 1285.65 \text{ K} \quad \text{and} \quad c = 2 \text{ 1/mm}^2. \quad (45)$$

The constant c in the exponent of (45.1) has the dimension $1/\text{m}^2$. Substitution of this function into the general solution (38) and evaluation of the integrals gives

$$\begin{aligned} \sigma_{rr} &= \frac{1}{2\sqrt{2\pi}} \frac{\alpha E T_0}{R^2 r^2 c} \left[r^2 (1 - e^{-cR^2}) - R^2 (1 - e^{-cr^2}) \right], \quad \sigma_{\varphi\varphi} = \frac{1}{2\sqrt{2\pi}} \frac{\alpha E T_0}{R^2 r^2 c} \left[r^2 (1 - e^{-cR^2}) + R^2 (1 - e^{-cr^2}) - 2R^2 r^2 c e^{-cr^2} \right], \\ u_r &= \frac{\alpha}{4cr\sqrt{\pi}} \left[(1+\nu)(T_0\sqrt{2} - T_0\sqrt{2}e^{-cr^2} + 2T_1cr^2\sqrt{\pi}) + (1-\nu)r^2 \frac{T_0\sqrt{2} - T_0\sqrt{2}e^{-cr^2} + 2T_1cR^2\sqrt{\pi}}{R^2} \right]. \end{aligned} \quad (46)$$

In figure 3.3 the radial stress σ_{rr} , circumferential stress $\sigma_{\varphi\varphi}$ and the radial displacement u_r are plotted as a function of radius r .

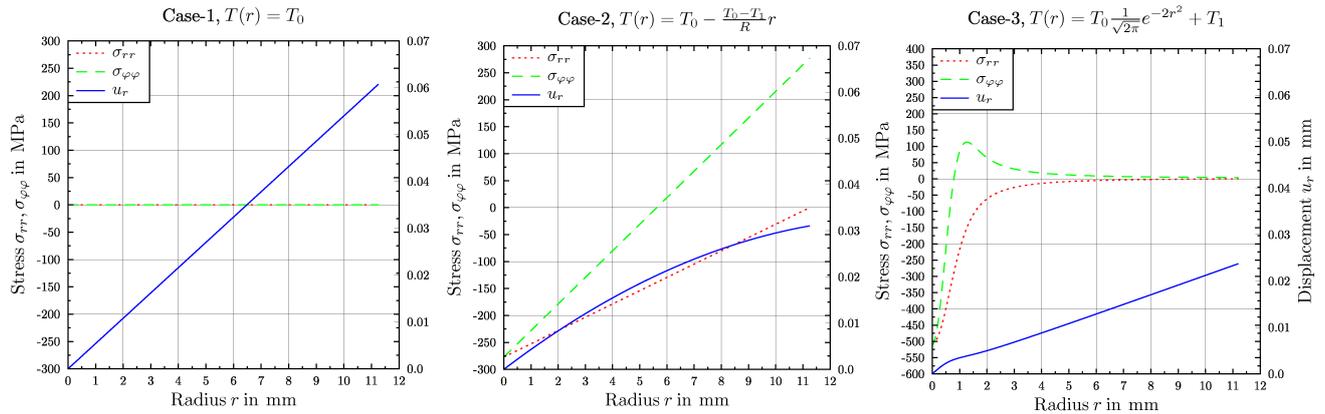


FIGURE 3

Analytical solution; radial stress σ_{rr} , circumferential stress $\sigma_{\varphi\varphi}$ and radial displacement u_r as a function of radius r for the three investigated temperature distributions $T(r)$

Numerical solution

The numerical model is analyzed by the *finite element method* (FEM). For the analysis the commercial Code *ABAQUS V6.6* is used. A detailed description of the functionality of this general purpose FEM code is given in [9]. The reason why the problem is solved by this method is to show the connection between numerical and analytical calculation tools in engineering education and engineering life. It is often possible to develop an analytical approximation for a problem. If such an approximation is available the numerical solution can be verified.

Finite element model

The symmetry of the circular domain is used. The model is meshed with CAX8T elements. This abbreviation means: continuum-type, 8-node axisymmetric thermally coupled quadrilateral, biquadratic displacement, bilinear temperature element. The boundary condition for the temperature is modeled via the ABAQUS user-subroutine *disp.f*. With this subroutine it is possible to define a temperature field dependent on the coordinates. The subroutine is written in *FORTRAN90* and the source code is available from the authors' website <http://institute.unileoben.ac.at/mechanik/>.

In the case of a constant temperature distribution there is no user subroutine necessary, but it is also implemented for convenience. The postprocessing of the numerical results was performed in ABAQUS-CAE, with the help of a global cylindrical coordinate system. This system has the same origin as the global Cartesian coordinate system. ABAQUS-CAE performs a coordinate transformation of the results from the Cartesian to the cylindrical system, and writes the data in tabular ASCII coded files. It is also possible to create contour and symbol plots or animations with this postprocessor. The authors use this functionality in their lectures about continuum mechanics, tensor calculus and finite element methods.

Case 1, $T(r) = \text{const.}$

In figure 4.1 the results from the numerical calculation for case 1 are shown. There is no observable difference between the analytical and the numerical solution.

Case 2, $T(r) = f(r)$

Figure 4.2 shows the results for case 2. The temperature varies linearly with the radius from the origin to a point on the circumference. For this temperature distribution the numerical computation gives also the same result as the analytical one.

There is no disagreement detectable between those two solutions.

Case 3, $T(r) = f(r)$

Figure 4.3 shows the result for case 3. In this case the temperature varying like a Gaussian function from the midpoint to a point on the circumference. A comparison of the analytical solution (figure 3) with the numerical one shows that there is also no identifiable disagreement between them.

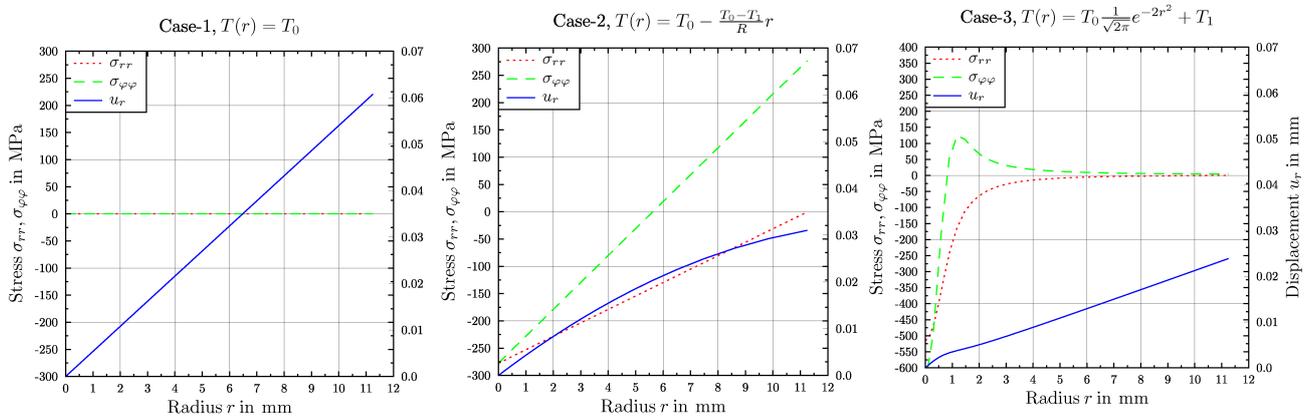


FIGURE 4

Numerical solution; radial stress σ_{rr} , circumferential stress $\sigma_{\phi\phi}$ and radial displacement u_r , as a function of radius r for the three investigated temperature distributions $T(r)$

Conclusions

1. The basic equations of linear thermoelasticity were developed and formulated in tensor notation, so they can be written in any coordinate system when the metric of this system is defined.
2. The basic equations of linear thermoelasticity were reduced to the *Lamé-Navier equations* specialised for cylindrical coordinates and written in physical components.
3. As an application of linear thermoelasticity, the problem of a thermally loaded cylindrical specimen was formulated and solved analytically. The basic equations of this problem were developed for plane strain and plane stress conditions. In this work the plane stress case was investigated. For three different boundary conditions the analytical solution was calculated and later compared with a numerical solution.
4. In this work a CAS is intensively used which takes care of all the time consuming calculations related to the tensorial formulation of the problem were done with it. It is clear that a lecturer should use such a tool if he is interested in presenting the concepts of tensor calculus and continuum mechanics. The necessary tensor operations are programmed in the CAS environment. Solutions of differential equations or evaluation of integrals are also performed with the CAS.
5. It is shown that with the aid of a CAS it is possible to examine different boundary conditions in very short time. Therefore students could get quickly an impression how equations and their solutions behave. One of the most striking benefits is that it is possible to solve different cases in a short time. The saved time that way can be used for discussions of the results in classroom.

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Derivation of the Lamé-Navier equation in cylindrical coordinates with *MAPLE*TM

The Lamé-Navier equation in cylindrical coordinates are derived in two different *MAPLE*TM worksheets. First the standard *MAPLE*TM packages are used. After the calculation of the basic quantities the special operators in the Lamé-Navier equation are developed and expressed in physical components. In the second worksheet the *MAPLE*TM vector calculus package is used. With this package it is possible to study the calculus of vectors in many different predefined coordinate systems. For example *Cartesian*-, *cylindrical*-, *spherical*-, *conical*-, *polar*-, *bipolar*-coordinates can be used. For details see the documentation [8]. It should be noted that the calculations are much shorter compared to the first approach, but it is not possible to look deep inside the mathematical and physical structure of the equations. However it depends on the application and the user which method is preferred. Another point concerning tensor calculus and *MAPLE*TM should be mentioned. In *MAPLE*TM there exists a special package for calculations with tensors. The package was developed to solve problems from the general theory of relativity. Therefore special routines are implemented to perform complex calculations in a short way. If a special metric is defined, it is possible to evaluate the *covariant metric tensor*, *contravariant metric tensor*, *determinant of metric tensor components*, *Christoffel symbols*, *Riemann tensor*, *Ricci tensor*, *Ricci scalar*, *Einstein and Weyl tensors etc.* by one command.

Excerpt of the *MAPLE*TM-worksheet

```

> #####
> # Lamé-Navier differential equation in cylindrical coordinates #
> #####
> restart;
> with(LinearAlgebra):
> # dimension of space, equations of the coordinate lines
> ndim:=3;

                                                                ndim:=3
> x[1]:=Theta[1]*cos(Theta[2]);
                                                                x1 := Θ1 cos(Θ2)
> x[2]:=Theta[1]*sin(Theta[2]);
                                                                x2 := Θ1 sin(Θ2)
> x[3]:=Theta[3];
                                                                x3 := Θ3

> assume(Theta[1],positive);
> assume(Theta[2],positive);
> # covariant basis #
> i:=0:
> for i from 1 to ndim do
< g[i]:= < diff(x[i],Theta[i]), diff(x[2],Theta[i]),
diff(x[3],Theta[i]) >;
> end do;

                                                                g1 := [ cos(Θ2)
                                                                sin(Θ2)
                                                                0 ]
                                                                g2 := [ -Θ1 sin(Θ2)
                                                                Θ1 cos(Θ2)
                                                                0 ]
                                                                g3 := [ 0
                                                                0
                                                                1 ]

> # covariant metric coefficients
> g_cov:=array(symmetric,1..ndim, 1..ndim);
                                                                g_cov:=array(symmetric, 1..3, 1..3, [])

> i:=0: j:=0:
> for i from 1 to ndim do
> for j from 1 to ndim do
> g_cov[i,j]:=simplify(DotProduct(g[i], g[j]));
> od;
> od;
> print('covariant metric coefficients=',g_cov);

                                                                covariant metric coefficients =, [ 1 0 0
                                                                0 Θ1^2 0
                                                                0 0 1 ]

```

The complete *MAPLE*TM worksheets can be downloaded from the authors' homepage <http://institute.unileoben.ac.at/mechanik/>. There is also a detailed derivation of the Lamé-Navier equations in general coordinates and especially in cylindrical coordinates available. Additionally the ABAQUS-input file and the Fortran coded User-subroutine is available there.