Numerical integration in tabular form

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Abstract — *The trapezoidal rule and Simpson's rule are common numerical techniques for approximate integration to be used when the function to be integrated* (a) *is given in analytical form, having an impracticable or unknown integral or* (b) *is given as values available only at a certain, limited number of points, usually equidistant — as will be dealt with in this study —, such as in tabular form, or, equivalently, values supplied by a computer program. We present simple formulas to make those rules easier when it is necessary to produce tables for lookup, typically calculated in a computer, tables from which graphs of the integral can also be made. Although these and other rules are generally taught everywhere in technological curricula, and are explained in innumerable books as applied to the integral from* a *to* b, *both given, we have found no description for the case where the abscissa varies in a range — say, from* a *to* x, *with* $a \le x \le b$ *—, which permits to build tables and is necessary, namely, to produce graphs. This problem appears also to be adequate for pedagogical purposes. We address the problem for equidistant points of the independent variable, with a description easy to transcribe into a computer language, and illustrate it with the computation of the volume of a spheroid. A mention is made to the approximate estimate of the errors through finite differences, in the absence of the analytical form of the integrand.*

Index Terms — *numerical integration, Simpson's rule, tabular form, trapezoidal rule.*

FUNDAMENTALS AND SCOPE

Numerical integration is a tool making it possible to compute a definite integral when the function, i.e., the integrand (considered here univariate), is not practicably integrable. This difficulty arises in one of the following cases: *(a)* the function, given analytically, has an impracticable or unknown integral; *(b)* the function is given as a set of values only for a certain number of points of the independent variable, usually equidistant points (as addressed in the present study), such as in tabular form, or values supplied by a computer program. The trapezoidal rule and Simpson's rule are common numerical techniques to solve the problem.

Although the aforementioned rules are generally taught everywhere in technological curricula, and are explained in innumerable books —for instance, the one by Greenspan and Casulli [1] or Hämmerlin and Hoffmann [2]—, the only case systematically studied is the integral from a to b, both given. Tables are then profusely constructed but only to show that these methods improve, tending to the correct value as the integration step decreases, with the typical limitations of machine precision and computing time. We have, thus, found no description for the case treated here, where the abscissa varies in a range, say, from a to x, with increasing x in that range, $a \le x \le b$. This not only permits to build tables for lookup —which is nowadays, in the computer age, usually no more indispensable—, but also is necessary to produce graphs. We think that this problem is interesting for pedagogical purposes, namely in technological curricula such as Engineering, Economics or Statistics. Although tables, such as the Gauss integral in Probability, are nowadays secondary in view of the availability of computers and calculators, the tables underlie the construction of graphs, and these have a definitive pedagogical value. We address the problem for equidistant points of the independent variable, and provide suitable transcription into computer language (pseudo-code), and illustrate its application with the computation of the volume of a solid of revolution, an oblate spheroid.

Examples for numerical integration are common: *(a)* the points in Table 1 (which are indeed from {1} with $a = 1$); or *(b)* the function given in {2}, which is the Gaussian probability density function. The unwieldy primitive of {1} was obtained through the "Integrator" [3] and is given in {3}. A complexity such as the one inherent in this case, or higher, may discourage the analytical resolution, pointing to a numerical method.

$$f(x) = \frac{x}{x^3 + a^3} \qquad \{1\}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \qquad \{2\}$$

$$F(x)=\ln\left[\frac{x^2-ax+a^2}{(x+a)^2}\right]+\left(2\sqrt{3}\right)\arctan\frac{2x-a}{a\sqrt{3}} \qquad \{3\}$$

x	$f(x)$
2.0	0.2222
2.1	0.2047
2.2	0.1889
2.3	0.1747
2.4	0.1619
2.5	0.1504

TABLE 1
TABULATED VALUES OF A FUNCTION FOR INTEGRATION

In order to compute definite integrals approximately, typical choices are, as mentioned, the trapezoidal rule and, more often, the Simpson's rule, both in the Newton-Cotes family of methods. When searching the literature for the trapezoidal or Simpson's rule in tabular form, innumerable sources are found, not, however, in the simple sense addressed here. Those sources typically do use tables for a different purpose: to show that progressively smaller integration steps provide better and better approximations (within the limits of computing time and computer arithmetic). Although this is an important aspect of these numerical integration techniques, here it will be considered that the integration step size (which will be kept constant) has been established, as may be obtained from analytical reasoning or from the equidistance of the values in a supplied table with the values of the function for a set of values for the independent variable.

The scope of this study is to present the result of integrating functions, a result to be shown in tabular form, both from integrands with an analytical expression and from values in a table, this latter form being of interest for illustration purposes or to produce tables and graphs for the integrated function. In what follows, formulas will be given for the trapezoidal rule, then for Simpson's rule, with an application to the computation of the volume of an oblate spheroid, followed by some conclusions.

TRAPEZOIDAL RULE AND SIMPSON'S RULE

The trapezoidal rule approximates the integral Y by a sum of trapezoidal areas

$$Y(a,b)=\int_a^b f(x)\,\mathrm{d}\,x\cong\left[\frac{1}{2}y_0+\sum_{i=1}^{n-1}y_i+\frac{1}{2}y_n\right]h \qquad \{4\}$$

with the usual definitions of $y_0=f(a)$, $y_i=f(x_i)$, $y_n=f(b)$, integration step $h=\dfrac{b-a}{n}$ and n number of steps. Introducing S,

$$S(x)=\frac{1}{h}Y(a,x) \qquad \{5\}$$

if $Y(a,x)$ —not $Y(a,b)$— is sought, for $a\le x\le b$, {4} becomes

$$S(x_k)\equiv S_k\cong\frac{1}{2}y_0+\sum_{i=1}^{k-1}y_i+\frac{1}{2}y_k \qquad \{6\}$$

remarking that only the values of x of the form $x_k=a+k\,h$ (k an integer) are needed. As n is usually large, it is convenient to write {6} in a recursive form, with the obvious value of $S_0=0$:

$$S_k\cong S_{k-1}+\frac{1}{2}\left(y_{k-1}+y_k\right) \qquad \{7\}$$

In a numerically simpler way, let $D=2\,S$:

$$D_k\cong D_{k-1}+\left(y_{k-1}+y_k\right) \qquad \{8\}$$

From {5} and the relation established between D and S, it is

$$Y\cong D\,\frac{h}{2} \qquad \{9\}$$

So, the procedure begins with computing y_0, and continues as follows:

$D_0 := 0$; $y_{\text{prev}} := y_0$
for $k := 1$ to n
$\{x := a + k * h$; $y := f(x)$; $D_k := D_{k\text{-}1} + y_{\text{prev}} + y$; $y_{\text{prev}} := y\}$
$Y := D * h / 2$

For Simpson's rule, the approximation to the integral $Y(a, b)$ is done by the following sum, with n even:

$$Y(a,b) = \int_a^b f(x)\mathrm{d}\,x \cong \left[y_0 + 4y_1 + \sum_{i=2(2)}^{n-2} (2y_i + 4y_{i+1}) + y_n \right] \frac{h}{3} \quad \{10\}$$

For this rule (in a manner parallel to the trapezoidal rule), from {10}, it is

$$S_k \cong S_{k-2} + y_{k-2} + 4y_{k-1} + y_k \quad \{11\}$$

and

$$Y \cong S\frac{h}{3} \quad \{12\}$$

Remark that the step is now double the one for the trapezoidal rule.

The procedure begins with computing y_0 and y_1, and continues as follows:

$S_0 := 0$; $y_{\text{m2}} := y_0$
for $k := 2$ to n step 2
$\{y_{\text{m1}} := f(a + (k - 1) * h)$; $y := f(a + k * h)$;
$S_k := S_{k\text{-}2} + y_{\text{m2}} + 4 * y_{\text{m1}} + y$; $y_{\text{m2}} := y_{\text{m1}}$; $y_{\text{m1}} := y\}$
$Y := S * h / 3$

APPLICATION

The above procedures are now applied to computing the volume of an oblate spheroid, i.e., an ellipsoid of revolution, as approximately the Earth, with polar radius, c, smaller than its equatorial radius, a. (Thus, with the axis of revolution vertical, the height is $2\,c$ and the width $2\,a$.) The formula of the spheroid is

$$\frac{x^2 + y^2}{a^2} + \left(\frac{z}{c}\right)^2 = 1 \quad \{13\}$$

Introducing $r = \sqrt{x^2 + y^2}$, this becomes

$$\frac{r^2}{a^2} + \left(\frac{z}{c}\right)^2 = 1 \quad \{14\}$$

The volume of a solid of revolution, V, can be computed by the well-known formula of {15},

$$V(z) = \pi \int_{-c}^{z} r^2(t)\mathrm{d}\,t \quad \{15\}$$

The volume of the oblate spheroid becomes

$$V(z) = \pi \int_{-c}^{z} r^2(t)\mathrm{d}\,t = \pi a^2 \int_{-c}^{z} \left[1 - \left(\frac{t}{c}\right)^2\right] \mathrm{d}\,t \quad \{16\}$$

or, with $x = t / c$, and introducing a dimensionless v,

$$v(z) = \frac{V(z)}{\pi a^2 c} = \int_{-1}^{z/c} (1 - x^2)\mathrm{d}\,x \quad \{17\}$$

So, the integrand is

$$f(x) = 1 - x^2 \quad \{18\}$$

immediately integrable to

$$F(x) = x - \frac{1}{3}x^3 \qquad \{19\}$$

Therefore, it is

$$v(z) = \frac{V(z)}{\pi a^2 c} = \frac{2}{3} + \frac{z}{c} - \frac{1}{3}\left(\frac{z}{c}\right)^3 \qquad \{20\}$$

This function goes from 0 (for $z/c = -1$) to 4/3 (for $z/c = +1$), as expected. Indeed, the volume of the (whole) spheroid is $V(c) = (4/3)\pi a^2 c$, with the particular value $(4/3)\pi a^3$ for the sphere of radius a.

i	z	$x = z/c$	$f(x)$	D	Y_{trap}	S	Y_{Simp}	Y
0	-2	-1	0	0	0	0	0	0
1	-1,6	-0,8	0,36	0,36	0,036			0,0373
2	-1,2	-0,6	0,64	1,36	0,136	2,08	0,1387	0,1387
3	-0,8	-0,4	0,84	2,84	0,284			0,2880
4	-0,4	-0,2	0,96	4,64	0,464	7,04	0,4693	0,4693
5	0	0	1	6,60	0,660			0,6667
6	0,4	0,2	0,96	8,56	0,856	12,96	0,8640	0,8640
7	0,8	0,4	0,84	10,36	1,036			1,0453
8	1,2	0,6	0,64	11,84	1,184	17,92	1,1947	1,1947
9	1,6	0,8	0,36	12,84	1,284			1,2960
10	2	1	0	13,20	1,320	20,00	1,3333	1,3333

TABLE 2
NUMERICAL INTEGRATION

As f was selected to be analytically integrable, it will be possible to compare the approximate integral with the exact values. Making $c = 2$, in Table 2 we have the values for the application of the trapezoidal and Simpson's rules, with $Y \equiv v$. In the table, with an arbitrary $h = 0{,}2$, according to the expressions previously deduced, for example for $i = 4$, it is:

From {18}: $f(x = -0{,}2) = 1 - 0{,}2^2 = 0{,}96$
From {8}: $D_4 = D_3 + y_3 + y_4 = 2{,}84 + 0{,}84 + 0{,}96 = 4{,}64$
From {9}: $Y_{\text{trap},4} = D_4\, h / 2 = 4{,}64\ (0{,}2 / 2) = 0{,}464$
From {11}: $S_4 = S_2 + y_2 + 4\, y_3 + y_4 = 2{,}08 + 0{,}64 + 4 \times 0{,}84 + 0{,}96 = 7{,}04$
From {12}: $Y_{\text{Simp},4} = S_4\, h / 3 = 7{,}04\ (0{,}2 / 3) = 0{,}4693$

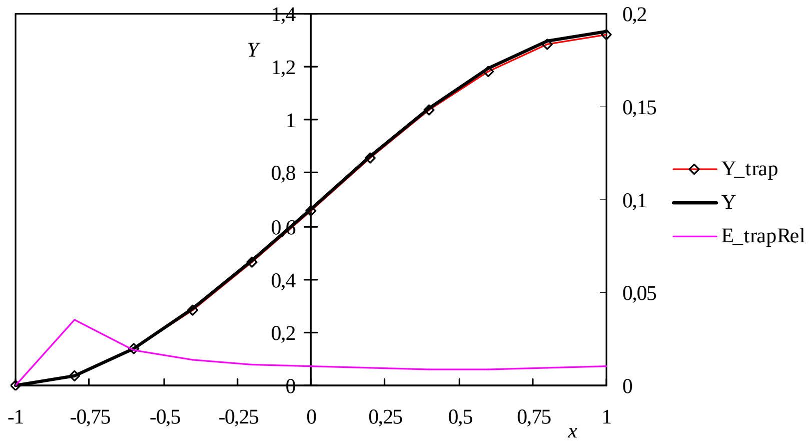

FIGURE 1
GRAPH OF Y_{TRAP} AND Y FROM TABLE 2

As regards the errors of the two estimates, trapezoidal and Simpson's rules, it is to be remarked that, with $h = 0{,}2$, the trapezoidal rule gives a final error of ~0,01 (i.e., |1,320–1,333|) against 0 for Simpson's rule, precisely 0 because the latter is exact for polynomials up to the second degree, as {18} shows is the case. In FIGURE 1 are shown the trapezoidal rule approximation (Y_trap), the exact result (Y, identical to the Simpson results), and the relative error (E_trapRel, shown as 0 for the initial value 0) reported to the right-hand side axis (small values), as functions of x.

The estimate of the error cannot be made from the analytical integrand, as already justified, because the derivative is supposedly not accessible. A solution is to compute an approximation to the derivatives, the 2.nd for the trapezoidal rule and the 4.th for Simpson's rule. This can be easily implemented by computing finite differences, and some results can be found on one of the author's webpages [4].

CONCLUSIONS

The common trapezoidal and Simpson's rules were presented, so as to facilitate the computation of a numerical integral in a tabular form, i.e., for the integral from a to x, with $a \le x \le b$, with a and b given, for equidistant points. This is convenient to obtain a table of the integral, useful both for lookup and for its graphical presentation. An application was made to the computation of the volume of a spheroid. The estimate of the errors is not possible, as an analytical form is considered unavailable, but can be obtained by finite differences.

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