Was Maxwell Aware of Including Relativity Theory in His Equations?

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Abstract — *Historically the field of electromagnetic fields is taught first by means of the classical theory of Maxwell. Classical electromagnetism is based on the well established Maxwell's equations. These set of four partial differential equations, together with the Lorentz force law are the complete set of laws of classical electromagnetism. Individually, the equations are known as Gauss's law, Gauss's law for magnetism, Faraday's law of induction, and Ampere's law with Maxwell's correction. The complete set was published in 1865 and since then James Clerk Maxwell is considered as one of the greatest scientist by numerous scientists. In 1905 Albert Einstein proposed the Special Theory of Relativity. Part of this theory is the law of electrodynamics which is based on the principle that the speed of light is the same for all inertial observers regardless of the state of motion of the source. This principle together with the Lorentz's transformation and coulomb's general law of force between charged particles, are the basis for calculating electromagnetic problems. In this Paper the electric field due to transient current waves is calculated. The transient current model is a step function Positive-Negative current wave model. This model is considered as the step response of an arbitrary transient. By convolution this response, any arbitrary transient current shape can be calculated. The electric fields are first analyzed and calculated by means of Maxwell's equations. Then, the same electric fields are calculated based on the Theory of Special Relativity and relativistic considerations. Comparing the results obtained by these two approaches shows that the calculations based on relativity theory are with full agreement with the ones obtained by Maxwell's equations.*
These results yield to a possible conclusion that that may be the answer to the above question, which is that Maxwell's equations for calculating the electromagnetic field indeed include the relativistic aspects of charges that travel in relativistic velocities although these relativistic considerations did not exist at his time.

Index Terms — *electromagnetic transient analysis, electromagnetic transient propagation, electromagnetic radiation, lightning, relativistic effects, switching.*

INTRODUCTION

Calculating the electromagnetic field due to transient currents in the time domain is important for assessing the influence of rapid changing currents such as: switching lightning, spikes, etc., on electric and electronic systems.
There are various approaches to address and teach electrodynamics and especially the field due transients. Extensive work has been done on the subject in the last 40 years. Analytical solutions for the electromagnetic fields based on Maxwell's equations were developed by different researchers. Uman and McLain [1] calculate the magnetic field due to various assumed forms of lightning channel current as a function of time. Expressions for the far field in the time domain and computer solutions for the total near and far fields are presented in later research by Uman, McLain and Krider [2]. This work is based on an antenna model. Other techniques for calculating the electromagnetic fields due to lightning are the monopole and dipole techniques presented by Uman and Rubinstein [3] and Thottappillil and Rakov [4]. An interesting analysis of the electromagnetic field from a vertically placed and moving square wave and a wire carrying uniform line charge are discussed by Rubinstein and Uman [5] and later on by Thottappillil, Uman and Theethayi [6]. Time-domain analysis is important for deeper understanding of the transient phenomenon. Two approaches of the time-domain analysis of the electric fields due to a step function model representing the lightning channel are reviewed and compared by Thottappillil and Rakov [7]. One is the Lorentz condition approach and the other is the Continuity equation approach. A complementary effect to those discussed above is an important effect of retardation. The origin of this effect is the well-known Lienard–Weichert potential to a single point charge. This idea was developed to the so-called F-factor analyzed for different cases by Thottappillil, Uman and Rakov [8].
Another calculation method is based on a wave-pair model presented by Braunstein[9]. This time-domain model describes the lightning wave as a step function, and the field is calculated directly from Maxwell's equations, with the addition of the so-called retarded potentials.
In this paper, the validation of Maxwell's equations dealing with systems which are under relativistic rules is analyzed by using the abovementioned wave-pair model [9]. The calculation is done for a step function. Thus, for any transient wave shape the response can be calculated by using the convolution integral.
A wave-pair step function is assembled of a dual step functions traveling to opposite directions where one wave is positive charged and the other is negative charged. The electric field due to this model is determined by using Maxwell's equations and then determining the solution for the same model by the using the Special Relativity Theory and

relativistic concepts in order to calculate the 'velocity field' and the 'acceleration field' [10]. These fields are the basic elements required for calculating the total field resulting from the current wave-pair model [11]. The results from both methods are compared. Comparison of the two methods yields identical results. This result yields to the conclusion that Maxwell's equations self contain the relativistic consideration that the Special Relativity Theory of Einstein is based on and therefore there is another method of teaching these fields according to relativistic considerations. The theory might be difficult but the formulation is much easier than the one according Maxwell's Equations.

THE PHYSICAL MODEL

The model presented in this paper is a general model for calculating the induced electric field due to an arbitrary transient current by convolution of the step response of the system. In this case the source is defined as a semi-infinite thin charge filament (current wave) that is cylindrically-symmetric, [9]. The medium around the source is free space.

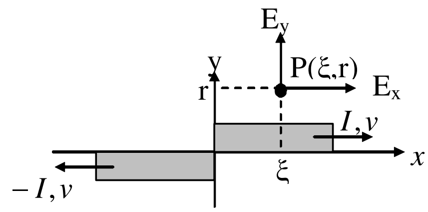

FIGURE 1
THE OPPOSITE POLARITY TWO WAVE MODEL

The model represented here consists of two step-current waves starting at the origin. The origin can represent the bottom of a cloud in the case of lightning down going discharge or an arbitrary place on a conductor when transient current occurs. The first one is a positive polarity wave traveling in the positive x-direction at a velocity of . The second one is a negative polarity wave traveling in the –x direction, at a same velocity of , see Fig.1. Due to this configuration the model is called an NP (Negative- Positive) or PN(Positive-Negative for the complementary model) step current wave.
Analyzing the response of the system to a step function is justified by the fact that if the induced electric fields due to a step function is known, and then by using the Convolution Theory, the response to any other function form (i.e. a typical real transient current wave such as switching or lightning) can be obtained.
Due to the fact that the source of the transient phenomenon is not defined, a NP (Negative- Positive) step wave-pair is used, as shown in Fig.1, where the negative polarity current wave moving in the opposite direction is required in order to satisfy the boundary conditions of the problem (this will be proved in the next section) and to maintain charge conservation. Thus, the necessary definition of the source is avoided. This wave-pair model is a one that is with total agreement with the conservation of the charge and therefore this model is suitable in the case of the source of the transient currents in systems containing indefinable source.
The electric field is calculated at an observation point *P(ξ, r)* as a function of time, *t*. The step function represents moving charges, where each charge is stationary at the origin prior to *t=0*. At *t=0*, the charges are infinitely accelerated to velocity , and then continue to travel with a constant velocity *v*.

THE SOLUTION BY MEANS OF MAXWELL'S EQUATIONS

The method for calculating the induced electric fields due to lightning stroke via relativity approach is compared with the results via Maxwell's equations. Therefore, a review of the calculation method through Maxwell's equations is presented next. Maxwell's equations for free space are:

$$\left.\begin{array}{l}\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\\ \nabla\times\vec{H}=\vec{J}+\dfrac{\partial\vec{D}}{\partial t}\\ \nabla\bullet\vec{D}=\rho\\ \nabla\bullet\vec{B}=0\end{array}\right\}\qquad(1)$$

Where the vector potential $\vec{A}$ is defined, based on the fourth term of (1):

$$\vec{B}=\nabla\times\vec{A}\qquad(2)$$

from (1) and (2):

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \tag{3}$$

Based on the above mentioned equations, together with Ohm's law, the wave equations for V and $\vec{A}$ are obtained:

$$\left.\begin{aligned} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \end{aligned}\right\} \tag{4}$$

These equations are valid only when the following boundary condition (the Lorentz gauge condition) is fulfilled:

$$div\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \tag{5}$$

In a cylindrical system, where all conductors are usually thin, it is worthwhile to introduce λ, the charge per unit length, instead of ρ, and the current I, instead of $\vec{J}$. The general solutions for the scalar and vector potential equations, in a point $P(\xi, r)$, which lies at a distance ρ' from an infinitesimal conductor element ds (see Fig.2), are:

$$\left.\begin{aligned} V &= \frac{1}{4\pi\varepsilon_0}\int_s \frac{\lambda(s, t - \frac{\rho'}{c})}{\rho'} \cdot ds \\ \vec{A} &= \frac{\mu_0}{4\pi}\int_s \frac{I(s, t - \frac{\rho'}{c})}{\rho'} \cdot d\vec{s} \end{aligned}\right\} \tag{6}$$

These are the well-known "retarded potentials".

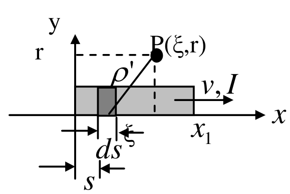

FIGURE 2
A CURRENT WAVE TRAVELING TO THE RIGHT OF THE x -AXIS AT A VELOCITY OF v

Calculating the potentials due to a traveling step function charge wave (see Fig.2) yields:

$$\left.\begin{aligned} V &= 30I\frac{c}{v}\left[\ln\left(U + \sqrt{U^2 + R^2}\right) - \ln(-\xi + \sqrt{\xi^2 + r^2}\right] \\ \vec{A} &= 30I\frac{1}{v}\left[\ln\left(U + \sqrt{U^2 + R^2}\right) - \ln(-\xi + \sqrt{\xi^2 + r^2}\right] \cdot \hat{\xi} \end{aligned}\right\} \tag{7}$$

Where v is the velocity of the charge wave propagation, ξ and r are the horizontal and vertical distances of the observation point from the origin, $\hat{\xi}$ is a unit vector in the x direction and the following definitions are used:

$$\begin{aligned} U &= \frac{c}{c+v}(vt - \xi) \\ R &= \sqrt{\frac{c-v}{c+v}} \cdot r \end{aligned} \tag{8}$$

By Substituting (7) into (3), the horizontal and vertical components of the electric field strength can be obtained:

$$\left.\begin{aligned}\vec{E}_{\xi} &= 30I\frac{c}{v}\left[\frac{\frac{c-v}{c}}{\sqrt{U^2+R^2}}-\frac{1}{\sqrt{\xi^2+r^2}}\right]\\ \vec{E}_{r} &= 30I\frac{c}{v}\cdot\frac{1}{r}\left[\frac{U}{\sqrt{U^2+R^2}}+\frac{\xi}{\sqrt{\xi^2+r^2}}\right]\end{aligned}\right\} \qquad (10)$$

(10) is the electric field strength components at the point $P(\xi, r)$, due to a current wave traveling on the x-axis with the velocity of v.

The vector and scalar potentials determined by (7) should be checked to fulfill the boundary condition of (5). In the case of a single current wave as described above, see Fig.2, (5) is not satisfied. The reason for that is that the source of the current wave has not been taken into account. Thus, the physical picture is distorted.

Trying various types of wave models leads to the conclusion, that the only topography of step function waves which satisfies the boundary condition of (5) is N-P or P-N wave-pairs. This was stated in previous section and shown in Fig.3.

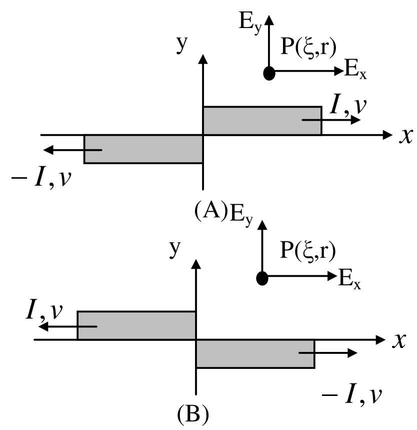

FIGURE 3
(A) A N-P WAVE-PAIR (B) A P-N WAVE-PAIR

Solving the potential equations for a N-P or a P-N model gives solutions which satisfy (5). These potentials are calculated in the same manner (7) was derived. Then, the potentials are substituted in (5) to obtain the electric field strength $\vec{E}$ of an N-P wave pair and the solutions are:

$$\left.\begin{aligned}\vec{E}_{\xi} &= 30I\frac{c}{v}\left[\frac{c-v}{c}\left(\frac{1}{\sqrt{u_1^2+R^2}}+\frac{1}{\sqrt{u_2^2+R^2}}\right)-\frac{2}{\sqrt{\xi^2+r^2}}\right]\\ \vec{E}_{r} &= 30I\frac{c}{v}\cdot\frac{1}{r}\left[\frac{c-v}{c}\left(\frac{u_1}{\sqrt{u_1^2+R^2}}-\frac{u_2}{\sqrt{u_2^2+R^2}}\right)+2\frac{\xi}{\sqrt{\xi^2+r^2}}\right]\end{aligned}\right\} \qquad (11)$$

Where:

$$\begin{aligned}u_1 &= \frac{c}{c+v}(vt-\xi)\\ u_2 &= \frac{c}{c+v}(vt+\xi)\\ R &= \sqrt{\frac{c-v}{c+v}}\cdot r\end{aligned} \qquad (12)$$

THE SOLUTION BY SPECIAL RELATIVITY THEORY

The Sub Model- Graphical Presentation

A step function can be observed as the case, in which charges are stationary at the origin and then at t=0 , they are accelerated at an infinite acceleration to a constant velocity of *v*. Thereafter, their velocity is constant. Due to this acceleration, the electric field strength at any observation point cannot be determined by using the Special Relativity Theory only, as it deals with cases, in which the charges move with constant velocity only. Moreover, from the observer's point of view, one part of the current wave travels at a constant velocity and the other part of the current wave is stationary at the origin. Furthermore, a solution by means of the Special Relativity Theory uses transformation of the problem to an alternative coordinate system, by Lorentz transformation. In this new coordinate system the source is stationary and the observation point travels in a constant velocity, the electric field strength is determined by using Coulomb's law. Then, the coordinates are transformed back to the original system by the reverse Lorentz transformation. In the presented case, the model has two current waves traveling in opposite directions. Therefore, it is impossible to find a coordinate system in which the conditions for using Special Relativity Theory are satisfied.

One way of solving this problem is by a direct way in which the field of a charge moving at a constant velocity ("The velocity field") and the field of an accelerating charge ("The acceleration field") are determined prior to the calculation of the electric field induced by the charges of the complete model. This method requires quit complicated physics [11].

A much more convenient method of getting the electric field strength at the observation point $P(\xi,r)$ by means of the Special Relativity Theory, is a method based on sub models. The model of Fig.1 is combined of sub models in such a manner that all parts of the model are either stationary or traveling at a constant velocity. Thus, the electric field strength is calculated for each part (sub model) by Special Relativity Theory and then by superposition theory the sum of the electric field strengths of all parts yields the desired solution. See Fig.4.

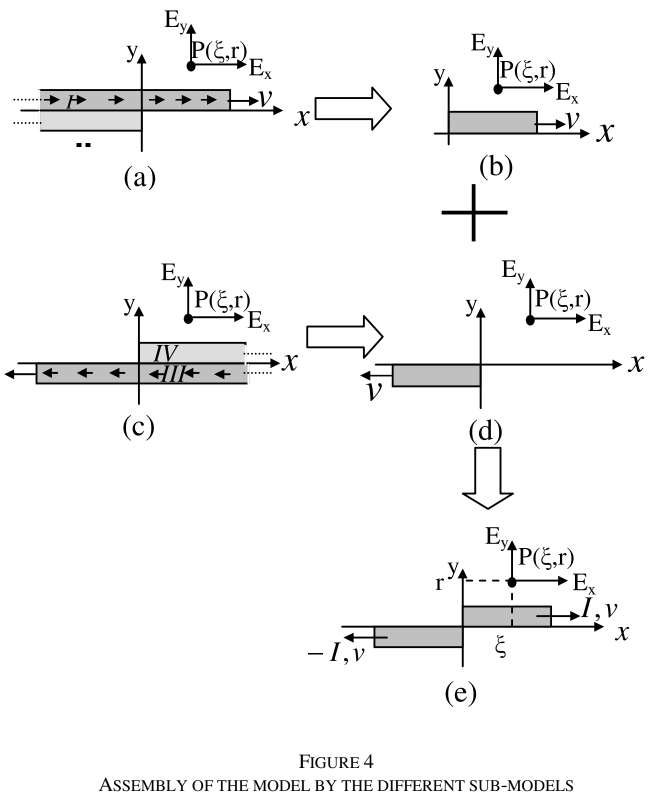

FIGURE 4
ASSEMBLY OF THE MODEL BY THE DIFFERENT SUB-MODELS

In Fig.4a the current wave, marked as I, consists of charges originating at infinity on the negative side of the x-axis. These charges move at a constant velocity v towards the positive direction of the x-axis as shown. Section II is a static part of a charge density, with an opposite polarity, located between infinity, on the negative side of the x-axis, and the origin. The sum of the electric field strengths at the observation point P(ξ,r) is the same as the electric field strength calculated due to the current wave described in Fig.4b. The same situation is described in Fig.4c and Fig.4d for a current wave of opposite polarity, traveling in the opposite direction. The electric field strength due to the total model (see Fig.4e) is the net sum of the field strengths due to the current waves described in Fig.4b and Fig.4d.

In Fig.4a and Fig.4a the influence of the current wave "tail" (from the origin to infinity) is eliminated by adding the field strength due to the uniformly moving charges and the field strength due to the opposite polarity static charges. This operation has been justified, both physically and mathematically, in [11].

The Sub Model- Mathematical Calculation

Constructing the lightning wave model by the sub-models, which consist of charges traveling at a constant velocity, v, only and segments of static charge densities, results in that the calculations of the electric field strength due to each part of the model are simplified. It is done either by the Special Relativity Theory for the traveling waves, or by calculating field strengths of static charge densities distributions. Next the calculations are presented. The electric field strength of the model shown in fig.4 is calculated in sections as follows.

1. ***The electric field strength of the wave traveling from -∞ to*** X_1: In Fig.4a the section marked as *I* is a current wave traveling at constant velocity v. The calculation of the electric field strength at the observation point $P(\xi, r)$ is done by transforming the coordinates at hand into a new coordinate system. In this new system, the traveling wave is stationary and the observation point is traveling in the opposite direction, according to Lorentz transformation. Thus, the electric field strength is calculated by Coulomb's law and then transformed back by the inverse Lorentz transformation to the original coordinate system, [10].The resulting electric field strengths at the observation point $P(\xi, r)$ are:

$$\left.\begin{aligned} \vec{E}_{XI} &= k\lambda\int_{-\infty}^{X_1}\frac{\gamma(\xi-x)dx}{\left[\gamma^2(\xi-x)^2+r^2\right]^{3/2}} = k\lambda\frac{1}{\gamma\left[\gamma^2(X_1-\xi)^2+r^2\right]^{1/2}} \\ \vec{E}_{YI} &= k\lambda\int_{-\infty}^{X_1}\frac{\gamma r dx}{\left[\gamma^2(\xi-x)^2+r^2\right]^{3/2}} = k\lambda\frac{\gamma(X_1-\xi)}{r\left[\gamma^2(X_1-\xi)^2+r^2\right]^{1/2}} \end{aligned}\right\} \qquad (13)$$

When, X_I is the horizontal location of the front of the current wave and γ is the lorentz factor that is:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad (14)$$

2. ***Cancellation of the wave's "tail" from -∞ to the origin***: A calculation of the electric field strength at the observation point $P(\xi, r)$ due to a static charge density of $-\lambda$, located on the x-axis from -∞ to the origin (see Fig.4a II) is added to the total field strengths presented by (13). This is done in order to cancel the influence of the electric field strength, due to the part of the traveling wave in the region of -∞ to the origin. The solution yields the electric field strength due to a current wave traveling in the positive direction of the *x*- axis, as shown in Fig.4b.The electric field strengths due to the above-mentioned static charge density are:

$$\left.\begin{aligned} \vec{E}_{XII} &= -k\lambda\int_{-\infty}^{0}\frac{(\xi-x)dx}{\left[(\xi-x)^2+r^2\right]^{3/2}} = -k\lambda\cdot\frac{1}{\sqrt{\xi^2+r^2}} \\ \vec{E}_{YII} &= -k\lambda\int_{-\infty}^{0}\frac{r dx}{\left[(\xi-x)^2+r^2\right]^{3/2}} = -k\lambda\cdot\frac{1}{r}\cdot\frac{\xi}{\sqrt{\xi^2+r^2}} \end{aligned}\right\} \qquad (15)$$

Thus, the total electric field strengths of the traveling current wave see Fig.5b are:

$$\left.\begin{aligned} \vec{E}_X &= \vec{E}_{XI}+\vec{E}_{XII} = k\lambda\frac{1}{\gamma\left[\gamma^2(X_1-\xi)^2+r^2\right]^{1/2}} - k\lambda\frac{1}{\sqrt{\xi^2+r^2}} \\ \vec{E}_Y &= \vec{E}_{YI}+\vec{E}_{YII} = k\lambda\frac{\gamma(X_1-\xi)}{r\left[\gamma^2(X_1-\xi)^2+r^2\right]^{1/2}} + k\lambda\cdot\frac{1}{r}\cdot\frac{\xi}{\sqrt{\xi^2+r^2}} \end{aligned}\right\} \qquad (16)$$

3. **The electric field strength of the total model:** A similar calculation to the former one, demonstrated in the last section, can be done for the current wave traveling in the negative direction of the x-axis. The total electric field strengths due to the wave shown in Fig.4d are:

$$\left.\begin{aligned}\vec{E}_X &= \vec{E}_{XIII} + \vec{E}_{XIV} = k\lambda \frac{1}{\gamma\left[\gamma^2(X_1+\xi)^2 + r^2\right]^{1/2}} - k\lambda \frac{1}{\sqrt{\xi^2+r^2}} \\ \vec{E}_Y &= \vec{E}_{YIII} + \vec{E}_{YIV} = -k\lambda \frac{\gamma(X_1+\xi)}{r\left[\gamma^2(X_1+\xi)^2 + r^2\right]^{1/2}} + k\lambda \cdot \frac{1}{r} \cdot \frac{\xi}{\sqrt{\xi^2+r^2}}\end{aligned}\right\} \qquad (17)$$

The solution for the total field strength of the model, presented in Fig. 4e is obtained by adding the field strength calculated at the observation point due to the positive traveling current wave, given by (16) and the field strength calculated for the negative charge density traveling in the negative direction of the x-axis (17). The total electric field strengths are:

$$\left.\begin{aligned}\vec{E}_X &= k\lambda\left(\frac{1}{\gamma\left[\gamma^2(X_1-\xi)^2 + r^2\right]^{1/2}} + \frac{1}{\gamma\left[\gamma^2(X_1+\xi)^2 + r^2\right]^{1/2}} - \frac{2}{\sqrt{\xi^2+r^2}}\right) \\ \vec{E}_Y &= k\lambda \cdot \frac{1}{r}\left(\frac{\gamma(X_1-\xi)}{\left[\gamma^2(X_1-\xi)^2 + r^2\right]^{1/2}} - \frac{\gamma(X_1+\xi)}{\left[\gamma^2(X_1+\xi)^2 + r^2\right]^{1/2}} + \frac{2\xi}{\sqrt{\xi^2+r^2}}\right)\end{aligned}\right\} \qquad (18)$$

Substituting the Lorentz factor (14) into the obtained results of (18) yield that these electric fields are identical with the electric field strengths calculated by using Maxwell’s (11).

CONCLUSIONS

In this paper, the electric field due to a step function wave-pair model is determined by using Maxwell's equations as well as Special Relativity Theory. This field is the basic element required for calculating the total field resulting from a transient time domain current wave. The resulting induced field's strength obtained from Relativity Theory is with full agreement with the results obtained by using Maxwell’s equations. The identity in the results lead to the following conclusions: a) Maxwell's equation are indeed self contained and include relativistic aspects and this is about 30 years prior the publication of the Special Relativity Theory of Einstein. b) Assuming that Maxwell's equations are correct another proof for Relativity Theory is shown.

Taking into consideration that all the above is validated in addition to our great appreciation to Einstein one can only appreciate Maxwell more for the greatness of forming equation that are so self contained and even include relativistic aspects that were not known in those days.

Moreover, the methods shown in this paper give to alternatives for teaching transient electric fields. The one by Maxwell is a straight forward method that leads to complicated partial differential equations. The second is a method through Special Relativity Theory which is a more complicated method in theory but the formulation is very simple.

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