Contextual Learning in Math Education for Engineers

Authors:

Jennifer Czocher, The Ohio State University, jczocher@math.ohio-state.edu
Greg Baker, The Ohio State University, baker@math.ohio-state.edu

Abstract — *Engineering faculty at The Ohio State University are disappointed in the inability of engineering students to understand and use differential equations in their advanced engineering courses, despite completing a required course in differential equations in the Department of Mathematics. The premise underlying the traditional math course is that students need first to master the abstract view of differential equations, prior to understanding or appreciating their applications in engineering. Instead, a new approach has been designed that (1) incorporates the solution to a differential equation as part of the analysis of engineering applications and (2) consistently treats core concepts. This study compares the treatment of a specific topic between a standard textbook and a set of class notes that support the new design.*

Index Terms — *Conceptual Knowledge, Differential Equations, Textbook Analysis, Undergraduate Mathematics Education*

Introduction

The study of differential equations (DEs) is a critical component of engineering and physical science majors. As with most mathematical topics, the curriculum for DEs is designed and coordinated by mathematicians. As a consequence, the presentation of key material is often couched in abstractions and general cases. That is, the topic is communicated in the language of mathematics and with few points of contact with the physical world. Research suggests that the manner in which a concept is presented influences which features of the concept the students find salient. For example, lecturers from differing disciplines tend to privilege one interpretation of "derivative" over others, and students' knowledge bases reflect their instructors' preferences [1].

The primary function of a service course in DEs is to prepare nonmath majors to use mathematics in the pursuit of knowledge in their home disciplines. However, the material is often presented in a simplified form, as a collection of special equations solvable by particular techniques. Furthermore, the standard curriculum is found wanting a stronger link to the physical origins of the equations being solved. By paying attention to the equations' natural context, the theory of DEs can be bound together into a conceptually coherent whole [2]. Engineering scholars suggest that this goal is accomplished by incorporating an engineering, or discipline-specific, perspective into mathematics curricula [3, 4]. Thus, teaching engineering students effectively, and especially for the transfer of knowledge, may require the forging of more explicit links between theoretical and applied perspectives.

The language of mathematics is often characterized as syntax, symbols, formalism, and abstraction. However, communication is accomplished through the use of examples. Exercises, problems, definitions, and theorems are examples as well as the more familiar textbook examples, nonexamples, and counterexamples. Thus, the choices made by the instructor, as well as the textbook, as to which examples, and in which sequence, the students should be exposed is of utmost importance. Well-chosen examples can highlight the salient features of a mathematical idea whereas other choices would obscure the important concepts [5].

Commercial curricula available at the K-12 level feature a wide array of instructional materials, resources, and pedagogical advice. In contrast, at the university level, the curriculum tends to be identical to the textbook. For mathematics lecturers, the textbook acts as a source of homework and exam problems; for teaching assistants, the textbook acts as a reference guide; for students, however, the textbook comprises a list of theorems, definitions, and formulae to be applied while completing assigned homework problems. Furthermore, the textbook may act as a surrogate instructor when the student is working problems or studying on his own. Since the textbook is central to many university mathematics courses, attention must be paid to the implicit messages it conveys to students and to instructors [6, 7, 8].

The present study examines and compares two textbooks on DEs, Boyce & DiPrima (8th Edition) [9] and Baker (n.d.) [10], that are currently in use at The Ohio State University (OSU) in a course for engineering and physical science majors. The course covers direction fields, first-order DEs, second-order linear DEs, systems of first-order

DEs, partial DEs, Fourier series, and applications. We focus only on the portions of each textbook in use at OSU. This exploration of the curricula is part of a larger effort to investigate the efficacy of each in teaching DEs to nonmathematics majors.

Literature Review

Many textbooks adhere to a common outline: exposition-examples-exercises. The exposition presents the topic to be studied, and the examples provide paradigmatic models for the students to apply to the exercises [11]. In fact, many exercises in the text do not even require the students to digest the material in the exposition and examples. [6] classified exercises in common undergraduate calculus textbooks according to the level of reasoning required in seeking a solution. He discovered that more than 90% of the problems in each text surveyed could be solved without attending to the intrinsic mathematical properties of the task. Moreover, in 70% of the exercises, students needed only to search through the solved examples. That is, students need only search the text for a similar example and copy its solution with appropriate modifications. In many cases, there is a disconnect between the tasks assigned from the end-of-chapter problems, and the material from which the tasks are taken. For example, a common task in calculus textbooks consists of determining whether a function is continuous or discontinuous. In many such problems, one does not need the formal definition of continuity, but the students are expected to produce solutions that involve copious amounts of fine-grained computations that have little to do with the properties of continuity [7]. Indeed, students may learn to answer a question correctly without having to think about why their answer is sufficient or why the question is important.

Textbook tasks may be phrased in predictable ways, and students pick up on these cues to help them arrive at solutions. Students have very concrete expectations about the relevance of information provided in the problem, the relevance of context to the solution, the intended difficulty of the questions, and past experiences [12]. The danger lies in the fact that problems derived from the situations in which engineers use mathematics are not well-defined. Naturally, one becomes concerned with how students may generalize and transfer their learning to other, messier, situations.

For students, and for instructors and researchers, constructing a meaningful understanding of a mathematical topic is a difficult task. Constructing a versatile and robust understanding of mathematics that is widely applicable and readily accessible in multiple settings is exponentially more difficult. One of the most overlooked components of mathematical understanding is students' coordination of formal and informal ideas. Since mathematics is perceived as a deductive system with formal rules of logic and syntax, it is presented to students linearly and hierarchically. Students expect that when the instructor or syllabus moves to a new topic, that the old topic will be left behind. Formal definitions and analytic solution techniques tend to overwhelm informal and intuitive notions [7], especially when the student is immensely successful at using a syntactic reasoning structure [13]. However, a flexible and mature understanding of mathematics is facilitated by coordinating formal and informal ideas.

Instead of constructing a well-integrated and well-connected understanding, students tend to compartmentalize their knowledge [14]. Compartmentalization refers to inconsistent or incoherent mathematical behaviors that manifest as the apparent lack of structure in the student's knowledge base. The compartmentalization of knowledge results in a fragmented collection of rules, procedures, definitions, and theorems which cannot be brought to bear on problems outside of the setting in which they were learned. For example, students learn the formal definition of function, but may not use it when classifying relations as functions [14] or when generating examples of functions [15].

Compartmentalization characterizes the mismatch between *concept image* and *concept definition* within the student's cognitive structure [16]. A concept definition is a verbal or written definition that describes or explains the concept in a non-circular way that is accepted by the mathematical community. The student's cognitive processes, experiences, mental models, and connections associated with the concept compose the concept image. That is, the concept image comprises the mental associations that determine the meaning of the concept to the student. A robust concept image is associated with the rich connections characteristic of conceptual knowledge [17]. Textbooks can either support or impede these connections. Tasks that encourage higher-order thinking engage students in cognitively demanding tasks such as interpreting, managing resources, constructing meaning, and using a flexible approach [18].

The textbook author makes certain assumptions about his readership and about the instructor enacting the curriculum described by the text. These assumptions include how to communicate mathematics to the reader and in what forms mathematics should be consumed. Some authors write so as to provide a "specific, correct path through the subject matter" whereas others act "more like a general guide or critical friend" who provides resources for the reader to generate his own concepts [11, p. 388]. Additional decisions belonging to the author include setting the level of cognitive difficulty and determining the prominence of motivational factors such as nonmathematical applications.

In developing and executing these features, the authors must bring into consideration the audience of the text and the purpose of the course it serves. These assumptions surface as implicit messages to the reader about the nature of mathematics and mathematics learning [8]. In undergraduate mathematics courses, the textbook might be the closest approximation to mathematics-in-practice to which the students are exposed. Hence, we must examine critically the features of the textbook that may influence student understanding.

Conceptual Framework

Our underlying assumption is that the textbook influences the manner in which the student comes to think about the mathematical topics he is studying. The goal of the textbook is to communicate mathematics to the student and to provide support to the instructor. Thus, the pedagogical choices made by the textbook authors must be examined in light of what material is included, excluded, and how it is presented.

We define the cognitive demand of a task as cognitive processes in which students must engage in order to complete a task [18, 6]. We classified tasks as requiring lower-level demands or higher-level demands using the framework developed by [19]. Lower-level tasks include *memorization* and performing *procedures without connections.* Memorization engages the student in the reproduction of previously encountered material. There is no connection to concepts or to meaning. In memorization tasks, there is no procedure applied, the solution is just a regurgitation of a fact. Tasks that require procedures without connections are characterized by a lack of ambiguity in the problem statement and an emphasis on obtaining a correct answer. These problems are typically solved through applying an algorithm without needing to understand how the procedure is connected to the underlying concepts. In particular, the correct procedure is evident from prior experience, the placement of the task, or is specifically called for in the problem statement. These lower-level problems can be solved by recall or by a search-the-text strategy, as in [6].

Higher level demands require a sustained level of effort, such as tasks that require the use of procedures with connections to concepts and tasks that engage the student in *doing* mathematics. Tasks that require procedures with connections may have solutions that coordinate multiple representations such as symbols and diagrams. These tasks are characterized by suggesting pathways to follow that are broad general procedures, but that are transparently connected to the underlying conceptual ideas. It is important to note that solutions to procedures-with-connections tasks will generally follow a procedure, but that the procedure cannot be applied mindlessly. *Doing mathematics* requires complex and nonalgorithmic thinking that is not explicitly called for by the task or worked out in an example. Tasks that engender mathematical practices require students to access and employ relevant knowledge and experiences, without being told which ones are relevant, and require students to analyze the task and examine constraints.

We extended and adapted the framework to analyze the expositional text and also to classify passages according to whether the passages gave high- or low-level explanations (exposition) or modeled high- or low-level tasks (examples). High-level expositions make connections to other mathematical topics, either within or external to the subject matter, or to physical interpretations. Low-level explanations are isolated and are disconnected from other subject matter. For example, a low-level explanation of a rule would refer only to another rule. A high-level explanation of that rule would also include a reference to another concept, context, informal or formal notion. Similarly, a high-level example would draw on many concepts and procedures, whereas a low-level example would display an algorithm.

Methods

We investigated the development of a single topic, solutions to nonhomogeneous ordinary DEs, within each textbook. We built a profile of the topic in each text by examining (1) the concept definition, (2) how the concept is developed, (3) the cognitive level at which the topic is treated, and (4) the epistemological messages conveyed by the text.

We selected the texts because they are both in use at OSU, and this report presents but one component of a study underway to compare learning outcomes of students enrolled in each model. In this report, we focus on the treatment of nonhomogeneous ordinary DEs, which are central to the study of DEs. General solutions to nonhomogeneous linear equations are constructed as a sum of the solution to the associated homogeneous equation and a particular solution to the nonhomogeneous equation. The general solution will contain arbitrary constants that can be set in order to satisfy constraints such as initial, boundary, or normalizing conditions on the function.

In Boyce & DiPrima [9], solutions to nonhomogeneous equations are discussed in sections 1.2 (Solutions of Some Differential Equations), 2.1 (Linear first-order Equations with Variable Coefficients), 2.2 (Separable first-order Equations), and 3.6 (Nonhomogeneous Second-Order Equations with Constant Coefficients; Method of Undetermined Coefficients), although they are present throughout the text, for example, in section 3.9 (Forced Vibrations). In

Baker's text [10], nonhomogeneous equations are not segregated from other equations and so solutions to nonhomogeneous equations are treated heavily throughout. The most direct instruction occurs in Chapter 1 sections 1.1 (Growth and Decay), 1.2 (Forcing Effects), 1.3 (Second-Order Equations: Growth and Decay), 1.4 (Second-Order Equations: Oscillations), and 1.5 (Forcing Terms: Resonances). As opposed to the assigned sections of Boyce & DiPrima, the Baker model includes nonhomogeneous boundary conditions in the study of partial DEs.

We employed a text-analysis approach to generate descriptions of how each text treats nonhomogeneous equations. Each textbook is available in portable document format (.pdf). Instead of looking only at the sections where nonhomogeneous equations are discussed explicitly, we used the search feature of the .pdf viewer to locate instances of general, homogeneous, and particular solutions within the context of solving nonhomogeneous equations. We did not count instances since (1) we only examined the chapters used in our course at OSU and (2) [9] often uses equation numbers instead of words. In addition, we carefully read each textbook section and worked the exercises in order to gain an idea of what the students might experience.

We originally classified each instance according to the role it played (definition, example, exercise, theorem, or exposition) to get a sense of how the terms were used. Many cases could not be classified in a single category since, for example, exposition could appear within a worked example. Since we were interested in the ways in which each textbook developed the concept, the instances were then coded according to whether connections to other concepts were made and what connections to other concepts were possible, both in and outside of mathematics, noting any inconsistencies. We then examined the instances through the lenses identified in the literature search: reasoning structures, cognitive difficulty, concept definition/concept image, and epistemological grounding.

Results and Discussion

First, we communicate the goals of each textbook in order to familiarize the reader with their purposes. Then we will address the guiding themes for each textbook in turn. Boyce & DiPrima represents a "standard" course on DEs. The intention of the authors, as stated in the preface, is for the text to be "widely useful" for undergraduate students of mathematics, science, or engineering through "a sound and accurate (but not abstract) exposition of the elementary theory of DEs with considerable material on methods of solution, analysis, and approximation that have proved useful in a wide variety of applications" [9, p. vii]. Baker's text could be classified as "reform oriented." However, unlike other reformation efforts [20], the immediate goal is not to restructure the classroom nor to involve multimedia commitments. It includes many of the same topics as the OSU implementation of Boyce & DiPrima, but the two do not share organizational features. Baker's text is intended for science and engineering majors, and uses common problems in science and engineering fields as the motivation for creating and for solving DEs. The text employs a modeling approach and its end goal is to shed light on "behavior such as the long time pattern, stationary or steady, its stability and the transition from some initial state. Subordinate to this is the identification of important length and time scales" [10, p. ii].

Textbook [9]

The authors define homogeneous equations in multiple ways: (1) an expression in which the left hand side is a function of the independent variable, the dependent variable and derivatives of the dependent variable with respect to the independent variable and the right hand side is zero, (2) an equation of the form $dy/dx = f(x,y)$ where f can be described as a function of the ratio y/x, and (3) the left hand side of a nonhomogeneous equation. Of the three definitions, (1) is used most frequently, but the three are never shown to be interchangeable. Nonhomogeneous equations are given in examples and exercises, but are not defined until the discussion on second-order linear equations with constant coefficients. Nonhomogeneous equations and linear equations, respectively, take the forms

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

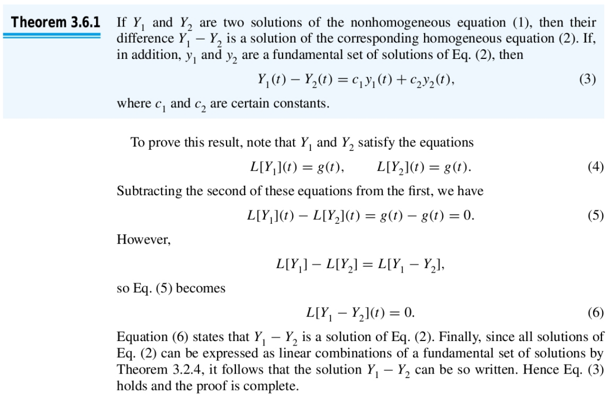
$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y. \tag{2}$$

Note that equation 1 does not have constant coefficients, despite the fact that the discussion takes place in a section on equations with constant coefficients. Of more immediate consequence is that equation 2 does not reflect any of the properties of linearity that are used in the construction of general solutions to nonhomogeneous equations. The discussion is limited to second-order equations, but the definitions of homogeneous, nonhomogeneous, and linear are extremely limited. The choice to write the definitions in these forms is arbitrary and the forms do not lend themselves easily to building a mathematical foundation for future learning nor to developing intuition about the structure of equations. There is no motivation given for why linearity is a useful definition, or a useful concept, nor why we choose to designate nonhomogeneous equations.

The term "particular solution" is defined informally as referring to "some solution $Y(t)$ of the nonhomogeneous equation" (Section 3.6, Method of Undetermined Coefficients). Prior to this usage, the text uses the phrase colloquially to refer to any specific solution of any equation. The authors do not speak of "homogeneous solutions," but rather "solutions to homogeneous equations." A general solution is one that represents all possible solutions of the DEs. It is also frequently identified as a solution that contains arbitrary constants. Sometimes the term refers to the solution of the associated homogeneous equation. In reference to the solution of a system of linear algebraic equations, the phrase "most general solution of the homogeneous system" corresponding to the nonhomogeneous system indicates that "general-ness" is a quantitative attribute. Thus, informal and formal definitions are inconsistent in their usage throughout the text.

[9] is segmented and modular and many of the chapters are written to be independent of the others. This allows for teacher flexibility in covering information in different orders, depending on the needs of the course constituency. Sections devoted to applications are segregated from, and typically follow, sections devoted to technique. Each module covers one solution technique or one application. The example and problems in each section represent the types of equations that can be solved with that technique. For example, Section 2.1 (Linear Equations with Variable Coefficients) exposes students to equations that can be solved by the method of integrating factor.

Theorems are stated in precise mathematical terms, in symbolic language and often resting on the theory of linear operators, as in Figure 1. The theorem in Figure 1 is proven syntactically, but many theorems in the text are justified through proof-by-example. The theorem is used to make the proof of the form of the general solution trivial. Several important connections are missed in this theorem: (1) only *one* particular solution is required to form the general solution, (2) the importance of linearity in the scope of the theorem, and (3) the impetus for subtracting the solutions.



Theorem 3.6.1 If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (2). If, in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (2), then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \qquad (3)$$

where c_1 and c_2 are certain constants.

To prove this result, note that Y_1 and Y_2 satisfy the equations

$$L[Y_1](t) = g(t), \qquad L[Y_2](t) = g(t). \qquad (4)$$

Subtracting the second of these equations from the first, we have

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0. \qquad (5)$$

However,

$$L[Y_1] - L[Y_2] = L[Y_1 - Y_2],$$

so Eq. (5) becomes

$$L[Y_1 - Y_2](t) = 0. \qquad (6)$$

Equation (6) states that $Y_1 - Y_2$ is a solution of Eq. (2). Finally, since all solutions of Eq. (2) can be expressed as linear combinations of a fundamental set of solutions by Theorem 3.2.4, it follows that the solution $Y_1 - Y_2$ can be so written. Hence Eq. (3) holds and the proof is complete.

FIGURE 1
BOYCE & DIPRIMA THEOREM 3.6.1

It is interesting to note that nonhomogeneous equations are presented in Chapters 1 and 2, which study first-order equations. For treating those cases, [9] introduces different solution techniques (method of integrating factor, method of separation). Indeed, the authors do not mention that the method of undetermined coefficients is applicable in some of these cases as well. The development of the method of undetermined coefficients proceeds results-first. This might be a point of confusion for students since there is no *a priori* reason to expect the solution to take the form of the right hand side of the equation.

An entire section (3.9, Forced Vibrations) is devoted to applying the method of undetermined coefficients to models of physical systems. The only physical context used is the damped, forced mass-spring system. The homogeneous solution is described as the transient solution, and its primary purpose is to "satisfy whatever initial conditions may be imposed." Thus, the energy present in the system in the form of initial position and velocity is dissipated through damping. The solution becomes the response of the system with the external force. Resonance is described as when the frequency of the forcing function is the same as the natural frequency of the system. Only periodic forcing is evident in the exercises. These descriptions provide much needed intuition about how solutions behave, but are statements of fact rather than fully integrated intuitive notions that extend to non-physical problems. The modularity of the book contributes to low-level explanations since it divorces the conceptual interpretations from the procedures.

One of the most attractive features of this textbook is the volume of tasks it offers. Assigned problem sets may vary according to the lecturer's tastes and the course goals. The tasks present at the end of any section reflect a progression from very concrete and narrow apply-the-technique tasks (Figure 2(a)) to highly structured multi-stage tasks in which the student is asked to supply information beyond solving the DE (Figure 2(b)) to explicitly scaffolded theory-development tasks (Figure 2(c)). Now, from our vantage point, we can see that the exercise in Figure 2(c) is meant to link the theory of DEs to the theory of linear operators. In fact, viewed as a linear operator acting on the vector space of sufficiently smooth functions, the DE will partition the space into the kernel of the linear operator and the image of the linear operator. This isomorphism theorem of linear algebra is the mechanism which allows us to construct the general solution to a linear DE by summing a spanning set of homogeneous solutions and a spanning set of particular solutions. In the development of solutions to second-order linear nonhomogeneous equations, we can see a solution technique that is similar to that of solving a quadratic equation: through factoring, we reduce a second-order equation to two first-order equations. But what is presented to the students is a reduced set of low-level activities. The tasks lack motivation and do not require the students to explain or justify their reasoning. Other than graphing their solution against one obtained through technological means (or checking the answers in the back of the book), the student has little recourse in determining if his solution is viable.

In each of the assigned problems, the procedure is explicitly called for or is evident from the statement or structure of the task. Even in the multi-stage and theory-driven problems, the step-by-step instructions eliminate the sense-making aspects of the task [18]. The procedure is outlined explicitly in the section summary, but in this case, the solution path cannot be memorized since the precise solution depends on the type of nonhomogeneous term. Even though the items in Figures 2(b) and 2(c) are connected to long-term behavior and linear algebra, the text makes the connections *for* the student. Therefore, these tasks exhibit a low-level of cognitive difficulty and are “procedures without connections." Furthermore, there is little to these problems that is unique to DEs. The path to the solution is consumed by tedious algebra or the application of first- and second-semester calculus techniques.

Overall, the authors employ a top-down approach to *giving* students mathematics. The dominant metaphor is that mathematics is “covered," evidenced by the authors' intention to present a broad range of topics and expect a wide range of competencies. The text communicates the strong epistemological commitment that knowledge is derived through synthetic reasoning structures. Problems that are straightforward to execute are valued and the doing of mathematics is reinforced as practicing solution techniques to technical problems. This can result in students thinking that mathematics is complete and that all the solutions are known. Indeed, they may “form the impression that there is an enormous amount to know, and that experts already know it all, when what society wants (or claims to want) is that each individual learn to enquire, to weigh up, to analyse, to conjecture, and to draw and justify conclusions" [21, p. 107].

In each of Problems 1 through 12 find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 2y' + 5y = 3\sin 2t$
3. $y'' - 2y' - 3y = -3te^{-t}$
4. $y'' + 2y' = 3 + 4\sin 2t$
5. $y'' + 9y = t^2e^{3t} + 6$
6. $y'' + 2y' + y = 2e^{-t}$
7. $2y'' + 3y' + y = t^2 + 3\sin t$
8. $y'' + y = 3\sin 2t + t\cos 2t$
9. $u'' + \omega_0^2 u = \cos\omega t, \quad \omega^2 \neq \omega_0^2$
10. $u'' + \omega_0^2 u = \cos\omega_0 t$
11. $y'' + y' + 4y = 2\sinh t$ *Hint:* $\sinh t = (e^t - e^{-t})/2$
12. $y'' - y' - 2y = \cosh 2t$ *Hint:* $\cosh t = (e^t + e^{-t})/2$

In each of Problems 13 through 18 find the solution of the given initial value problem.

13. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
15. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
16. $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1, \quad y'(0) = 0$
17. $y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$
18. $y'' + 2y' + 5y = 4e^{-t}\cos 2t, \quad y(0) = 1, \quad y'(0) = 0$

(a) Apply the technique

Behavior of Solutions as $t \to \infty$. In Problems 30 and 31 we continue the discussion started with Problems 38 through 40 of Section 3.5. Consider the differential equation

$$ay'' + by' + cy = g(t), \qquad \text{(i)}$$

where a, b, and c are positive.

30. If $Y_1(t)$ and $Y_2(t)$ are solutions of Eq. (i), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?
31. If $g(t) = d$, a constant, show that every solution of Eq. (i) approaches d/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ also?

(b) Multi-stage tasks

32. In this problem we indicate an alternate procedure[7] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \qquad \text{(i)}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.
(a) Verify that Eq. (i) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.
(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (i) can be found by solving the following two first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

(c) Scaffolded theory development

Figure 2
Tasks from Section 3.6

The inconsistent and inaccurate usages of concept definitions may be a source of confusion for students, and may contribute to incorrect associations. For example, in applying the method of undetermined coefficients, the guess for the particular solution of a nonhomogeneous equation also contains unknown constants. Frequently using the term “general solution" to refer to a solution that contains arbitrary constants and using the same term to refer to the completed solution may prematurely trigger students to stop seeking other parts of the solution. The modular organization of the text, the authors' chosen modes of communication, and the level of cognitive demand fostered by the end-of-section tasks render visible, and reinforce, the authors' epistemological commitments to what constitutes mathematical knowledge. Ideas are not connected and the students do not have to decide how to approach a problem. Taken together, these considerations may contribute to the construction of a fragmented concept image and a low-level way of engaging in mathematics.

Textbook [10]

[10] is organized by context and follows an example-exposition-exercise format. In each section, a detailed discussion of one or two application-oriented examples is followed by an abstract mathematical view of the material. The text is integrated, rather than modular. While this structure improves the continuity of the text's narrative, it may prove more difficult for students to use it as a quick reference. Instead of theorems, the text gives general “principles." Similar to [9], some of the principles are proved and some are justified through examples, most often, the principles are consequences of the worked examples and they are always stated in language that is consistent throughout the text. This organic approach to mathematical reasoning may help students appreciate the need for abstract mathematical statements. In [10], a solution to a DE is any function that can be substituted into the equation and the equation still makes sense (Principle 2). Furthermore, solutions are educated guesses at functions that might make sense

when substituted into the equation. [10] does not serve to preoccupy the students with memorizing techniques and conditionals, but encourages them to make and justify guesses. Every solution is a conjecture until it is proven to make sense.

A nonhomogeneous equation is defined as describing a system in the presence of forcing. Particular and homogeneous solutions are defined concomitantly (p. 15):

> From a mathematical perspective, different forcing terms will produce different responses, and if the system is described by a DE, then this means we should expect different solutions to the DE. These solutions will be called particular or inhomogeneous solutions. There is still the evolution of the system from some initial state, which would occur even in the absence of forcing. This free response of the system is reflected in the presence of homogeneous solutions and we saw several examples in the previous section.

The definition imbues each type of solution with a character based on its role in satisfying the DE. The particular solution "takes care of" the forcing, but it cannot satisfy the initial conditions. Instead of defining the general solution as one with arbitrary constants, [10] presents the particular solution as being incapable of satisfying an initial value problem on its own. Another prominent feature of the above passage is that of setting expectations. Instead of stating why objects must or must not have certain features, the author communicates expectations. Phrases such as "we expect" or "we anticipate," followed by checking, encourage conjecture and sense-making. As a pedagogical tool, the author manages expectations to transform informal reasoning structures into formal ideas. Thus, the dominant metaphor of the text is "sense-making."

[10] forges a strong connection between the system's response to forcing and the particular solution of a nonhomogeneous equation - the response mimics the force. This connection is used to explain growth and decay, resonance, and the effects of boundary conditions in partial DEs. The text makes explicit that general solutions are constructed through linear combinations of a homogeneous solution (an element of the linear operator's null space, a response to initial conditions) and a particular solution (a representative of the equivalence class of functions mapped to the nonhomogeneous term, the system response to forcing). In the case of a two-point boundary value problem, such as $u'' + u = f(t)$ with $u(0) = u(L) = 0$, the eigenfunctions are described as independent homogeneous solutions of the associated homogeneous equation. A linear combination of all the eigenfunctions must be taken in order to get a general solution, which is the Fourier series of f. By immediately, and consistently, representing the relationship between particular, homogeneous, and general solutions, a natural foundation is laid upon which a deeper understanding can be built in more advanced theoretical courses.

The method of undetermined coefficients is introduced in Section 1.2 (Forcing Effects) for first-order linear equations in the context of a mixing problem. The procedure is developed in terms of making guesses and balancing coefficients. In contrast, [9] follows a highly algorithmic presentation in [9], in which the authors derive a formula for the solution to first-order linear nonhomogeneous DEs. It is possible to complete the homework problems in that section using only the ultimate formula. However, the formula in [9]'s development allows the authors to treat first-order nonhomogeneous DEs with variable coefficients. In [10] it is unclear what sorts of guesses extend to linear DEs with nonconstant coefficients. Nevertheless, conceptual understanding and sense-making are stressed in [10].

There are far fewer problems in [10], and they tend to be less technically difficult. Almost all problems are "word problems," but some task sequences do focus on skill acquisition. In comparison, the [9] exercises require far more attention to the details of integrating complicated functions. It is not unusual for tasks in [10] to admit solutions that are several pages in length. Typical "word problems" appear in multiple contexts ranging from chemical reactions, to objects falling with drag, to temperature gradients, and require students engage in complex, nonalgorithmic thinking. Consider the following excerpt:

> A cooling fin is designed to allow heat to conduct along a thin plate while heat is transport [sic] to the ambient air through Newton's law of cooling that states the flux of heat is proportional to the difference in temperature between the fin and the air...Instead of two initial conditions, we have one condition at $x = 0$ and one at $x = L$ (these conditions are called boundary conditions). Use these conditions to find the solution for the temperature profile. In particular, find the temperature at the end $x = L$. The parameter K depends on the material of the plate, whereas H depends on the exposed area of the plate. Estimate a value for HL^2/K that will make the temperature at the end of the plate close to T_0.

Figure 3
A high-level problem

This problem is messy. It requires students to deal with ambiguity, make assumptions, and tie the outcomes back to the context of the problem. Problem 3 requires the solution of a two-point boundary value problem, and the task is fully integrated into the mathematical context of second-order linear equations with constant coefficients. Thus, later discussions about boundary value problems or heat conduction in a rod may be built on the informal knowledge about temperature profiles built here. The key feature of these sequences of tasks is that they are related to the examples worked in the text, but are not identical. This fact, coupled with the varying contexts of the tasks, maintains a high level of cognitive demand.

Through detailed examples, the author makes assumptions as the solution develops, instead of in the problem statement. This serves two pedagogical purposes: detailing why certain mathematical formalisms are necessary and modeling problem solving behavior. Taken together, these features indicate that the author conceptualizes mathematics as sense-making and values intuition over technical competencies. Thus, [10] encourages forms of knowledge that enable students to approach complex problems and make decisions in the course of solving them. The concepts associated with solving a nonhomogeneous DE are built upon ideas with which the students are already familiar. Thus their informal and common-sense knowledge structures can be coordinated with the formal, and more versatile, mathematical language [22]. Embedding problems in contexts about which students have real, experiential intuition may help support mathematical language development and associate already-constructed, and newly constructed, mental images and knowledge structures with symbols. That is, the structure of the text may help students generate meaning for the syntax. Throughout [10], the problems increase in complexity, not just in technical difficulty, and so cognitive resources are consumed by higher-order tasks instead of by computation. In solving such problem sequences, students must recognize not only how, but also when, to use procedures and concepts when approaching a problem [23]. Since mathematical development is coordinated with physical intuition, the student's conceptual knowledge may be more richly connected and robust. Experiencing mathematics in context and building a more robust concept image may serve to facilitate transfer to other contexts.

Connections to Past and Future Research

[9] encourages a wide range of syntactical competencies, but engineering scholars and mathematics education researchers testify that this is not the sort of knowledge that is applicable in settings outside the mathematics classroom [24, 25]. Baker's textbook was constructed in response to criticisms of a course on DEs for engineering and physical science majors voiced by engineering faculty [26]. Increased cognitive difficulty, and rich tasks, are linked with increased learning [27], higher-order thinking skills [28, 29], persistence [30], and enjoyment of mathematics [31, 30]. However, proceduralizing information is not a negative behavior [32]. In fact, proceduralizing requires that the student notice and use patterns [33]. Systematic proceduralizing, done by an authority such as the textbook, may contribute to the compartmentalization of knowledge [34]. One direction for future research would be to investigate to what extent textbooks influence students' developing concept images.

Future research should also address the shortcomings of this study. We analyzed only the portions of the textbook that are in use at OSU in a course for engineers and physical science majors. The texts were searched using terms that are common in Baker's textbook. However, in Boyce & DiPrima, equations are often referred to by number instead of by a name. There may be many more instances of particular, homogeneous, and general solutions present in [9] than are recorded in this study. Indeed, we did not examine linguistics, reader cues, nor other similar features of textbooks that help students locate meaning within the written text. It should also be noted that the critique is conceptualized from a deficit model. The aim here was not to imply that Boyce & DiPrima has little merit, but to highlight some features that are present in a common textbook that are incompatible with research on the didactics of mathematics. Our purpose is to encourage others to examine critically the textbooks in use.

References

[1] Bingolbali, E. & Ozmantar, M., "Factors shaping mathematics lecturers' service teaching in different departments", *International Journal of Mathematical Education in Science and Technology*, 40, no. 5, 2009, pp. 597–617.

[2] Tall, D., "Lies, damn lies... and differential equations", *Mathematics Teaching*, 114, 1986, pp. 54–67.

[3] Abramovich, S. & Grinshpan, A., "Teaching mathematics to non-mathematics majors through applications", *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 18, no. 5, 2008, pp. 411–428.

[4] Pennell, S., Avitabile, P., & White, J., "An engineering-oriented approach to the introductory differential equations course", *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 19, no. 1, 2009, pp. 88–99.

[5] Rowland, T., "The purpose, design and use of examples in the teaching of elementary mathematics", *Educational Studies in Mathematics*, 69, no. 2, 2008, pp. 149–163.

[6] Lithner, J., "Mathematical reasoning in calculus textbook exercises", *The Journal of Mathematical Behavior*, 23, no. 4, 2004, pp. 405–427.

[7] Raman, M., "Coordinating informal and formal aspects of mathematics: Student behavior and textbook messages", *The Journal of Mathematical Behavior*, 21, 2002, pp. 135–150.

[8] Raman, M., "Epistemological messages conveyed by three high-school and college mathematics textbooks", *The Journal of Mathematical Behavior*, 23, no. 4, 2004, pp. 389–404.

[9] Boyce, W. E. & DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems*, 8th edn., New York: Wiley, 2009.

[10] Baker, G., "An introduction to differential equations for scientists and engineers", , Unpublished.

[11] Love, E. & Pimm, D., "This is so: A text on texts", in A. Bishop (ed.), *International Handbook of Mathematics Education*, Dordrecht, The Netherlands: Kluwer Academic Publishers, 1996, pp. 370–410.

[12] Crisp, V., Sweiry, E., Ahmed, A., & Pollitt, A., "Tales of the expected: The influence of students' expectations on question validity and implications for writing exam questions", *Educational Research*, 50, no. 1, 2008, pp. 95–115.

[13] Weber, K., "How syntactic reasoners can develop understanding, evaluate conjectures, and generate counterexamples in advanced mathematics", *The Journal of Mathematical Behavior*, 28, 2009, pp. 200–208.

[14] Vinner, S. & Dreyfus, T., "Images and definitions for the concept of function", *Journal for Research in Mathematics Education*, 20, no. 4, 1989, pp. 356–366.

[15] Dahlberg, R. P. & Housman, D. L., "Facilitating learning events through example generation", *Educational Studies in Mathematics*, 33, no. 3, 1997, pp. 283–299.

[16] Tall, D., *Concept Image and Concept Definition*, 1983, Utrecht, Netherlands: OW & OC, 1988, pp. 37–41.

[17] Hiebert, J. & Lefevre, P., *Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis*, Hillsdale, NJ: Lawrence Ealbaum Associates, 1986, pp. 1–23.

[18] Stein, M. K., Grover, B. W., & Henningsen, M., "Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms", *American Educational Research Journal*, 33, no. 2, 1996, pp. 455–488.

[19] Smith, M. S. & Stein, M. K., "Selecting and creating mathematical tasks: from research to practice", *Mathematics Teaching in the Middle Shcool*, 3, 1998, pp. 344–350.

[20] Rasmussen, C. & Kwon, O. N., "An inquiry-oriented approach to undergraduate mathematics", *The Journal of Mathematical Behavior*, 26, no. 3, 2007, pp. 189–194.

[21] Mason, J., "Asking mathematical questions mathematically", *International Journal of Mathematics Education in Science and Technology*, 31, no. 1, 2000, pp. 97–111.

[22] Engel Clough, E. & Driver, R., "A study of consistency in the use of students' conceptual frameworks across different task contexts", *Science Education*, 70, no. 4, 1986, pp. 473–496.

[23] Skemp, R. R., *The Psychology of Learning Mathematics, Expanded American Edition*, Hillsdale, New Jersey: Lawrence Erlbaum Associates, Inc, 1987.

[24] Carraher, T. N., Carraher, D. W., & Chielmann, A. D., "Mathematics in the streets and in schools", *British Journal of Developmental Psychology*, 3, no. 1, 1985, pp. 21–29.

[25] Rittle-Johnson, B. & Alibali, M., "Conceptual and procedural knowledge of mathematics: Does one lead to the other?", *Journal of Educational Psychology*, 91, 1999, pp. 175–189.

[26] Crowther, K., Thompson, D., & Cullingford, C., "Engineering degree students are deficient in mathematical expertise - why?", *International Journal of Mathematical Education in Science and Technology*, 28, no. 6, 1997, pp. 785–792.

[27] Silver, E. A. & Stein, M. K., "The quasar project: The "revolution of the possible" in mathematics instructional reform in urban middle schools", *Urban Education*, 30, no. 4, 1996, pp. 476–521.

[28] Glass, B., "Comparing representations and reasoning in children and adolescents with two-year college students", *The Journal of Mathematical Behavior*, 23, no. 4, 2004, pp. 429–442.

[29] Schoenfeld, A. H., "Metacognitive and epistemological issues in mathematical understanding", in E. A. Silver (ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Hillsdale, NJ: Lawrence Erlbaum Associates, 1985, pp. 361– 379.

[30] Middleton, J. A. & Spanias, P. A., "Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research", *Journal for Research in Mathematics Education*, 30, no. 1, 1999, pp. 65–88.

[31] Martin, J. H., Hands, K. B., Lancaster, S. M., Trytten, D. a., & Murphy, T. J., "Hard but not too hard: Challenging courses and engineering students", *College Teaching*, 56, no. 2, 2008, pp. 107–113.

[32] Peled, I. & Segalis, B., "It's not too late to conceptualize: Constructing a generalized subtraction schema by abstracting and connecting procedures", *Mathematical Thinking and Learning*, 7, no. 3, 2005, pp. 207–230.

[33] Engelbrecht, J., Bergsten, C., & Køagesten, O., "Undergraduate students' preference for procedural to conceptual solutions to mathematical problems", *International Journal of Mathematical Education in Science and Technology*, 40, no. 7, 2009, pp. 927–940.

[34] Schoenfeld, A. H., "When good teaching leads to bad results: The disasters of "well taught" mathematics courses", *Educational Psychologist*, 23, no. 2, 1988, pp. 145–166.