An Application of Matrix Diagonalization in Engineering: Stress Matrix

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Abstract — In this communication we present a teaching experience, in the framework of the EUROPA Project, which was carried out during the academic year 2001-2002 in the Higher Technical School of Industrial Engineering of the Polytechnic University of Valencia (Spain). A simple application of the theory of endomorphisms of Euclidean spaces to strength of materials is shown. This experience has involved a collaboration between a basic science department (Department of Applied Mathematics) and a technological one (Department of Mechanics of Continuous Mediums and Theory of Structures).

Index Terms — teaching experience, interdepartmental collaboration, endomorphism theory, stress matrix.

INTRODUCTION

The EUROPA Project, which is developed at the Universidad Politécnica de Valencia (Spain), incorporates initiatives in educational management, teaching, learning and subject content. It is directed as much at promoting and encouraging the active participation of the students in the educational process, as at the teacher training of the faculty. One of the aims of this programme is to encourage the introduction of new teaching-learning methods in the subjects given at our University.

Within this framework, the AMA–6 “Transversal collaboration in teaching” program, aims to improve communication between basic and technological subjects by swapping teachers from one course to another. The purpose of this is to familiarise the first year students with the real technological problems, thus giving meaning to the basic subjects whose inclusion in the degree course they do not understand. This clearly stimulates the motivation of the first year students at the same time as managing to give structure to the content of the degree course.

The educational experience set out in this paper was developed as part of the Industrial Engineering degree in the subjects “Linear Algebra”, taught by the Department of Applied Mathematics, and “Elasticity and Strength of Materials”, given by the Department of Mechanics of Continuous Mediums and Theory of Structures. These subjects are taught in the first and second years of the above mentioned degree course, respectively. This teaching initiative was developed during an hour long class of “Linear Algebra” and consisted of presenting two problems taken from the “Elasticity and Strength of Materials” syllabus. These were proposed to the first year students while teachers of these subjects from both the first and second years were present. The intention was to motivate the student with these problems, while at all times looking to use the terminology which a first year student associates with a degree course such as Industrial Engineering. Once they were solved, it was attempted to draw physical conclusions from the mathematical results.

The two chosen problems were aimed at a detailed study of an application of matrix diagonalization in engineering. The stress state within an elastic solid - that which recuperates its initial shape when the forces causing its deformation stop working - can be worked out if we know the stress matrix of each point of the solid. The stress matrix diagonalization allows us to obtain the principal stresses of a specific state stress and the calculation of the equivalent stress in agreement with a criterion of fracture and / or yield. In this case the Tresca and Von Mises yield state criteria were used. To go into the topic in more depth, references (1), (3) and (6) may be consulted.

DESCRIPTION OF THE METHODOLOGY

When considering a technical degree course, one of the problems arising from the teaching of basic subjects is how to contextualize the content within the general framework of the degree and in relationship with the technological subjects...
subsequently studied. This problem is not a simple one to tackle, when taking into account the limiting effect of the scarce technical preparation of the first year students. Nevertheless, this must not be used as an excuse to avoid the problem. Learning situations need to be presented to the students where it can be shown how the contents and the methods of the basic subjects can be applied to different problems which are linked to their professional future. In order for these experiences to be successful, the teacher should make them attractive enough to arouse the students’ interest, but at the same time not overcomplicated to the point that the students become unmotivated. The teacher should also choose the right moment for the experiences to be presented.

The method proposed by us consists of a practical session during a class of basic subject in which the students take part and two teachers, one from the basic subject and one from the technological one, are present. During this practical session, an engineering problem is developed, whose modelization and subsequent solving are carried out using tools from the basic subject which have been previously studied by the students. The session is carried out in the following way:

- The teacher of technological subject briefly presents the problem to be treated and describes the model to tackle it with.
- The teacher of basic subject emphasizes the results and methods of his/her speciality subject which are used to solve the problem, thus demonstrating the value of the subject as a tool. Then, specific situations related to the problem are presented to the students and they are helped by their teacher to transform them into problems which are common to basic subject according to the established model.
- The students solve these problems individually or guided by their teacher, and the results obtained are handed to the teacher of the technological subject.
- The solutions are interpreted from an engineering point of view by the teacher of the technological subject. He/she concludes the initial problem and finishes the session by reviewing and assessing it.

**MATHEMATICAL PRELIMINARIES**

In this section, we collect some basic topics and results that are used in the paper. They are taken from ([4] y [5]):

Let \( V \) be an \( n \)-dimensional vector space over a field \( K \) (\( P \) or \( X \)) and let \( T : V \rightarrow V \) be an endomorphism of \( V \).

**Definition** An element \( \alpha \in K \) is called an eigenvalue of \( T \) if there is a vector \( x \neq 0 \) in \( V \) such that \( T(x) = \alpha \cdot x \), each such vector is called an eigenvector of \( T \) for the eigenvalue \( \alpha \). The spectrum of \( T \) is just the set of all eigenvalues of \( T \). If \( \alpha \) is an eigenvalue of \( T \), then the kernel of \( T-\alpha I \) consists of all the eigenvectors for the eigenvalue \( \alpha \) and the zero vector. This subspace is called eigenspace of \( T \) for the eigenvalue \( \alpha \) and it is denoted by \( E(\alpha) \).

**Definition** \( T \) is \( K \)-diagonalizable, if \( V \) possesses a basis of eigenvectors of \( T \). Now, we consider \((V,\langle,\rangle)\) an \( n \)-dimensional Euclidean space over \( P \) and let \( T : V \rightarrow V \) be an endomorphism of \( V \).

**Definition** \( T \) is called symmetric endomorphism if \( \langle T(x),y \rangle = \langle x,T(y) \rangle \ \forall x,y \in V \).

**Theorem** An endomorphism \( T : V \rightarrow V \) is symmetric if and only if \( V \) possesses an orthonormal basis of eigenvectors of \( T \) with respect to which \( T \) is represented by a real diagonal matrix.

If \( A \) is an \( n \times n \)-matrix over a field \( K \) (\( P \) or \( X \)) and \( B \) is the canonical basis of \( V = K^n \), then there is a unique endomorphism \( T : V \rightarrow V \) such that the matrix representation of \( T \) with respect to the basis \( B \) is exactly the matrix \( A \). In this way it is possible to translate the above results into matricial language.

**APPLICATION: STRESS MATRIX**

**Introduction to the problem**

The classic approach to the study of bodies given by Mechanics supposes the invariability of the relative distances between the points of each body, thus adopting the model of a rigid solid or non-deformable body, which does not lose its shape. The aim of Elasticity and Strength of Materials, on the contrary, is to determine the stress fields and deformations of deformable bodies.

In order to find out the set of stresses which act on each point in a structure, this is analysed either by the classical methods or by modeling the structure using structure or finite elements computer programs. In this way we can establish which point or points of the structure that show the most deformities and, therefore, the point or points of the structure where the yield or fracture of the material originates. Figure 1 shows stress distribution in a cantilever beam with an applied point load on its free end. The colours indicate different levels of stress in each point of the structure. The most deformed area of the beam shows a gradient of stresses which can be seen visually due to its greater variation of colours. As we shall subsequently see, the mathematical formulation used for predicting a failure criterion associated to the yield or fracture is closely connected to stress matrix diagonalization.
In order to obtain the stress state of a point, a Cartesian coordinate system \(xyz\) is established, by considering an infinitesimal prismatic element of faces parallel to the coordinated planes at the point under study (Figure 2). The set of stresses acting on the point makes up its stress state and is uniquely defined when the nine stresses shown in Figure 2 are known. This set of stresses is set out as a matrix in the following way:

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix}
\]

This matrix thus defined, is called Cauchy stress matrix.

The \(\sigma_i\) terms of the diagonal are called normal stresses and, physically, they represent extensions or compressions, perpendicular to the faces of the differential element found at the point. The normal stresses are directly related to the lengthening or shortening of the sides of the differential element and, as a consequence, to the variation in its volume. The \(\tau_{ij}\) terms are called shear stresses and they are tangent to the faces of the element. Shear stresses are directly related to the distorsion or change of angles and shape of the element. When dealing with metallic materials, it is the distorsion energy, associated to this change in shape, which is the magnitude to be controlled and limited. This mechanics work which is used in the variation of shape is closely linked to the beginning of the yield of the material prior to fracture.

The stress matrix is always a real, symmetric matrix and is therefore diagonalizable with respect to an orthonormal basis of its eigenvectors. The \(\sigma_1, \sigma_2\) and \(\sigma_3\) values common to the stress matrix are called principal stresses while the vectors \(u_1, u_2\) and \(u_3\) are known as principal directions. Once the state stress acting on a point, referred to a randomly chosen system of \(xyz\) axes, is known, and the previous stress matrix is defined, it is possible to determine the stresses acting in any other direction of \(\mathbb{R}^3\). In this way, the direction of the space in which the crystalline slide takes place and which is associated to the yield of the material or the atomic rupture of the bonds, can be obtained. The formulation of failure criteria is used to predict the yield or fracture of the material. These criteria are commonly formulated using the principal stresses. When there is a state stress in which there only exist stresses perpendicular to the surfaces, the stress state is called principal stress state, (Figure 3). It can easily be seen that the stress matrix associated to a principal stress state is clearly a diagonal matrix:

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
\]

where its eigenvalues are ordered in the form \(\sigma_1 \geq \sigma_2 \geq \sigma_3\).

Given that the stress matrix is real and symmetric, it is thus diagonalizable and, therefore, a principal stress state can be obtained from any initial state. In other words, the principal stress matrix is obtained through the diagonalization of the Cauchy stress matrix which is associated with a point and, additionally, the principal stresses are always real numbers. As a result, for any point of the solid there always exists a reference system made up of three perpendicular axes with respect to which extensions or compressions are only observed on the faces of the differential element.

As we have already stated, once the values of the stress matrix are obtained and in order to quantify how dangerous a stress state is, failure criteria are used. In developing our teaching experience, we used two different criteria: Maximum Shear Stress Criterion or Tresca Criterion and Maximum Distorsion Energy Criterion or Von Mises Criterion. These criteria are formulated in the following way:

TRESCA’ S CRITERION [6]: \[
\sigma_{eq} = \sigma_1 - \sigma_3 \leq \sigma_{ult}
\]

VON MISSES’S CRITERION [6]: \[
\sigma_{eq} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} \leq \sigma_{ult}
\]

where \(\sigma_{eq}\) is called equivalent stress and \(\sigma_{ult}\) the permissible stress that must be supported by a material considered to be safe for the structure. Using these criteria, an equivalent stress can be determined which may be directly compared to data extracted from uniaxial standard tension tests carried out on specimens of the material (Figure 4). In this way, a comparison is set up based on the degree of danger of two uniaxial stress states, whose associated distorsion energy must be equal in a limit situation.

Practice

Now, we present one of the problems developed in the teaching experience:
Figure 5 shows the stress state of a point \( A \) of a steel structure, which is determined by the matrix:

\[
\begin{bmatrix}
\sigma_x & \sigma_y & \sigma_z \\
\sigma_y & \sigma_z & \sigma_x \\
\sigma_z & \sigma_x & \sigma_y
\end{bmatrix}
\]

Assume the permissible stress for the steel is \( \sigma_{\text{ult}} = 2600 \text{ kp/cm}^2 \). Find the maximum value of \( \sigma \) by applying the Tresca and Von Mises criteria.

**SOLUTION:**

- **Principal stresses:** In order to calculate the principal stresses we find the real roots of the characteristic polynomial of \( [\sigma] \):
  \[ \det ([\sigma] - \lambda[I]) = (3\sigma - \lambda)^2 \]
  then \( \sigma_1 = 3\sigma \), \( \sigma_2 = 0 \), \( \sigma_3 = 0 \).

- **Principal directions:** In order to determine the principal directions we begin calculating the eigenspaces \( E(3\sigma) \) and \( E(0) \)

\[
E(3\sigma) = L \left( \begin{array}{c}
1 \\
1 \\
1
\end{array} \right) = L(u_1), \\
E(0) = L \left( \begin{array}{c}
0 \\
\frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}}
\end{array} \right) = L(u_2, u_3)
\]

where \( L \) denotes the span of a subset.

Therefore an orthonormal basis of eigenvectors of \( [\sigma] \) is \( \{ u_1, u_2, u_3 \} = \left\{ \begin{array}{c}
1 \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}}
\end{array} \right\}, \left\{ \begin{array}{c}
0 \\
\frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}}
\end{array} \right\} \)

Figure 6 shows the stress state of the point \( A \) with respect to the principal axes.

Hence, we can write \( [\sigma] = P[\sigma_o]P^T \) where \( [\sigma_o] = \begin{bmatrix} 3\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and \( P = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \)

Figure 7 shows the stress state of the point \( A \) with respect to the Cartesian coordinate system and the principal axes.

Once we know the principal stresses, we can obtain the maximum value of \( \sigma \) applying the failure criteria:

- **Tresca’s Criterion:**
  \[
  \sigma_{\text{eq}} = \sigma_1 - \sigma_3 \leq \sigma_{\text{ult}} = 2.600 \Rightarrow \sigma = \frac{2.600}{3} = 866.7 \text{ kp/cm}^2
  \]

- **Von Mises’s Criterion:**
  \[
  \sigma_{\text{eq}} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = \sqrt{\frac{1}{2}[(3\sigma - 0)^2 + 0 + (0 - 3\sigma)^2]} = 2\sigma \leq \sigma_{\text{ult}} = 2.600 \Rightarrow \sigma = \frac{2.600}{3} = 866.7 \text{ kp/cm}^2
  \]

Any value of \( \sigma \) lower than 866.7 \text{ kp/cm}^2 does not cause the yield of the material.

**ASSESSMENT OF THE EXPERIENCE**

Transversal collaboration between basic and technological departments allows for the exchange of information and experiences which undoubtedly both professionally and personally enriches the teachers taking part. An enrichment which has brought about the possibility of this type of teaching experience that may be the first step towards a subsequent interdepartmental collaboration. Likewise, the students come to see how the subjects that go to make up their degree are connected and the very presence of these subjects in their study course is justified.

Furthermore, the opportunity is taken to introduce the first year student to mathematical modeling, showing them the role of mathematics in the development of science and technology. This, in turn, gives us the chance to solve real problems and promote certain skills such as the reflection on and valuing of the results obtained, to encourage critical awareness and to stimulate creative capacity, all of which will be very useful for the engineer of the future.

The carrying out of this type of experience does not adversely affect either the total number of hours assigned to the basic subject or its content, as most of the session is given over to solving examples common to the subject and to strengthening the content. The time needed for this experience was not overly long, about 60 minutes.

In our opinion, in order to achieve a wholly satisfactory result the teachers involved must be present during the entire session as this gives a greater flexibility and allows them to complement each other. Furthermore, this makes it possible to compare the different terminology used in both subjects to refer to the same concept. The stress matrix spectrum, for example, is also the set of principal stresses, the orthogonal matrix of change of basis is referred to as the axes rotation.
We would like to point out that it is not convenient to use overly complicated examples and tools when designing the experience, so that they are accessible to the students this experience is aimed at.

We have been able to verify that the valuation by the Higher Technical School of Industrial Engineering students taking the subject “Linear Algebra”, where the work sessions were carried out, has been very positive. At first our students were surprised at the unusual nature of the methodology, but then went on to become interested not only in obtaining the result but also in confirming that mathematical tools are useful for solving problems from other subjects included in their degree of a more engineering–based nature.

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REFERENCES


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