Pedagogic Comparison between Time and Frequency Domain formulations of Dynamic Analysis of Structures

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Abstract

The response of structural systems subjected to dynamic loads can traditionally be evaluated in time domain or in frequency domain. Solution methods based in frequency domain have been in development in recent years. The choice of the method depends on the physical properties of the system. In those with frequency-dependent physical properties or with hysteretic damping, frequency domain analysis is recommended. The frequency domain analysis has become common use only after the advent of the FFT (Fast Fourier Transform) algorithm developed by Cooley and Tukey [1], for substantially reduce the computational effort in evaluating the DFT (Discrete Fourier Transform), doing it competitive with the time domain methods. Thenceforth, the frequency methods have been receiving significant contributions in the search for efficiency. In its latest edition Clough and Penzien [2] present a detailed treatment of analysis in the frequency domain. Ferreira et al. [3] present an interesting comparison of the time and frequency formulations for dynamic analysis. Although time domain methods are easily assimilated by engineering students, the same does not happen to frequency domain methods, especially to civil engineering students. The goal of this article is to introduce a pedagogic comparison between these two important methods to the solution of structural systems subjected to dynamic loads, in order to make it easier for civil and mechanical engineering students to learn this subject.

1. Introduction

The response of a structure subjected to dynamic loads can be basically calculated by two ways: by solving equations of motion in time or frequency domain. When the system parameters are frequency-dependent, the procedure in the frequency domain is more suitable. As an example, one can cite the interaction case of soil-structure systems in which stiffness and damping forces are frequency dependent.

The frequency domain analysis has became of ordinary use only after the emergence of FFT (Fast Fourier Transform) algorithm, developed by Cooley and Tukey [1], because substantially reduced the computational effort in the DFT (Discret Fourier Transform) evaluating, making it competitive with the methods of time domain. Since that time, the frequency domain methods have been receiving significant contributions in search of its efficiency. Clough and Penzien [4], in their first edition with an excellent text about structural dynamics, presented some elements of structural analysis in the frequency domain. In their latest edition [2] there is a more detailed analysis in frequency domain.

2. Equation of motion

Consider a mass-spring system with one degree of freedom, subjected to a load p(*t*). The equation of motion is obtained from the resultant of forces acting on the mass, which are the very loading, the viscous damping force (proportional and opposite to the mass velocity), and elastic spring force (proportional and opposite to the displacement from the equilibrium position). If v(*t*) is the displacement at time *t*, *m* is mass, *c* is the damping coefficient, *k* is the spring elastic constant and p(*t*) is the loading at time *t*, then the resulting force is:

$$\sum \boldsymbol{F} = \boldsymbol{m}\ddot{\boldsymbol{v}}(\boldsymbol{t}) = \boldsymbol{p}(\boldsymbol{t}) - \boldsymbol{c}\dot{\boldsymbol{v}}(\boldsymbol{t}) - \boldsymbol{k}\boldsymbol{v}(\boldsymbol{t}) \;\; \boldsymbol{or} \;\; \boldsymbol{m}\ddot{\boldsymbol{v}}(\boldsymbol{t}) + \boldsymbol{c}\dot{\boldsymbol{v}}(\boldsymbol{t}) + \boldsymbol{k}\boldsymbol{v}(\boldsymbol{t}) = \boldsymbol{p}(\boldsymbol{t}) \qquad (1)$$

3. Steady-state response to a harmonic loading

3.1 Time Domain

The motion equation is a differential equation and its solution depends on the loading p(*t*). Obtaining solutions for the cases of loading sine and cosine, one finds the general solution to harmonic loading (i.e., a linear combination of sine and cosine with the same frequency). Thus, defining the natural system frequency as $\omega = \sqrt{k/m}$ and damping ratio as $\xi = \dfrac{c}{2m\omega}$, one can rewrite the motion equation as follows:

$$\ddot{\mathrm{v}}(t) + 2\xi\omega\,\dot{\mathrm{v}}(t) + \omega^2\,\mathrm{v}(t) = \frac{\mathrm{p}(t)}{m} \qquad (2)$$

Adopting the loading $\mathrm{p}(t) = b\,\mathrm{sen}\,\overline{\omega}t$ with angular frequency $\overline{\omega}$, and defining the ratio $\beta = \overline{\omega}/\omega$, one obtains the motion equation, which is given by:

$$\mathrm{v}(t) = \frac{b}{k}\left[\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\right]\left[(1-\beta^2)\,\mathrm{sen}\,\overline{\omega}t - 2\xi\beta\cos\overline{\omega}t\right] \qquad (3)$$

Likewise, for a loading $\mathrm{p}(t) = a\cos\overline{\omega}t$, the solution is:

$$\mathrm{v}(t) = \frac{a}{k}\left[\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\right]\left[2\xi\beta\,\mathrm{sen}\,\overline{\omega}t + (1-\beta^2)\cos\overline{\omega}t\right] \qquad (4)$$

And finally, when both loads act simultaneously, the general solution is:

$$\mathrm{v}(t)=\frac{1}{k}\left[\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}\right]\left\{[b(1-\beta^2)+2a\xi\beta]\operatorname{sen}\overline{\omega}t+[a(1-\beta^2)-2b\xi\beta]\cos\overline{\omega}t\right\} \quad (5)$$

3.2 Frequency Domain

One can take profit of the periodic complex exponential function properties [5] and apply a "complex loading" $\mathrm{p}(t)=be^{i\overline{\omega}t}$. Thus, the motion equation becomes:

$$\ddot{\mathrm{v}}(t)+2\xi\omega\dot{\mathrm{v}}(t)+\omega^2\,\mathrm{v}(t)=\frac{b}{m}e^{i\overline{\omega}t} \quad (6)$$

whose solution is:

$$\mathrm{v}(t)=\mathrm{G}(\overline{\omega})e^{i\overline{\omega}t} \quad (7)$$

where:

$$\mathrm{G}(\overline{\omega})=\frac{b}{k}\left[\frac{1}{(1-\beta^2)+i(2\xi\beta)}\right]=\frac{b}{k}\left[\frac{(1-\beta^2)-i(2\xi\beta)}{(1-\beta^2)+(2\xi\beta)^2}\right] \quad (8)$$

This response proved to be simpler, since it presents a single term. In time domain there are terms in sine and cosine.

4. Response to any periodic loading

4.1 Time Domain

Any periodic loading (i. e., not necessarily harmonic) can be expressed in the form of Fourier series:

$$\mathrm{p}(t)=a_0+\sum_{n=1}^{\infty}a_n\cos\overline{\omega}_n t+\sum_{n=1}^{\infty}b_n\operatorname{sen}\overline{\omega}_n t \quad (9)$$

with:

$$a_0=\frac{1}{T_p}\int_0^{T_p}p(t)dt\ ;\quad a_n=\frac{2}{T_p}\int_0^{T_p}p(t)cos\overline{\omega}_n dt\ ;\ \ b_n=\frac{2}{T_p}\int_0^{T_p}p(t)sns\overline{\omega}_n dt \quad (10)$$

The system's response to that loading is given by:

$$\mathrm{v}(t)=\frac{1}{k}\left(a_0\sum_{n=1}^{\infty}\left[\frac{1}{(1-\beta_n{}^2)^2+(2\xi\beta_n)^2}\right]\left\{[b_n(1-\beta_n{}^2)+2a_n\xi\beta_n]\operatorname{sen}\overline{\omega}t+[a(1-\beta^2)-2b\xi\beta]\cos\overline{\omega}t\right\}\right) \quad (11)$$

4.2 Frequency Domain

As follows, one also can express the loading as a complex Fourier series:

$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{i\bar{\omega}_n t} \tag{12}$$

with:

$$P_n = \frac{1}{T_p} \int_0^{T_p} p(t) e^{i\bar{\omega}_n t} dt \tag{13}$$

Nesse caso, encontra-se a resposta a cada componente da série, que valerá:

$$v_n(t) = H_n P_n e^{i\bar{\omega}_n t} \tag{14}$$

Where $v_n(t)$ is a complex parameter, which means a real vector rotating in the complex plane. H_n is defined as follows:

$$H_n = \frac{1}{k}\left[\frac{1}{(1-\beta_n{}^2) + i(2\xi\beta_n)}\right] = \frac{1}{k}\left[\frac{(1-\beta_n{}^2) - i(2\xi\beta_n)}{(1-\beta_n{}^2)^2 + (2\xi\beta_n)^2}\right] \tag{15}$$

The final response will be:

$$v(t) = \sum_{n=-\infty}^{\infty} H_n P_n e^{i\bar{\omega}_n t} \tag{16}$$

Where $v_n(t)$ is real now, due to the presence of conjugate pairs.

5. Response to a general loading

5.1 Time Domain

Consider the overall loading $p(t)$ shown in Figure 1, below:

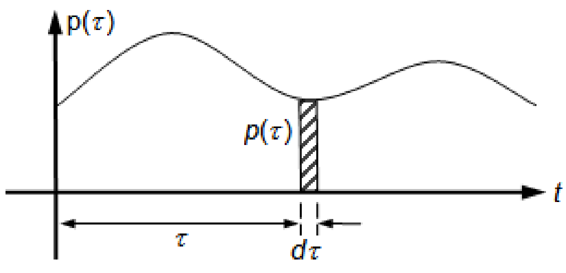

Figure 1 - Impulsive Differential Load $d\tau$ of an Overall Loading $p(t)$.

Based on the approximate response of an impulsive load $p(\tau)d\tau$, one can get the response to that loading, that will be:

$$v(t) = \int_0^t p(\tau)\, h(t-\tau) d\tau \tag{17}$$

This integral is a convolution integral (BRIGHAM, 1974), and the function $\mathrm{h}(t-\tau)$ is the unit impulse response, defined by:

$$\mathrm{h}(t-\tau)=\frac{1}{m\omega_0}\operatorname{sen}\omega_D(t-\tau)e^{-\xi\omega(t-\tau)} \tag{18}$$

where:

$$\omega_D=\omega\sqrt{1-\xi^2} \tag{19}$$

is the damped frequency of the mass-spring system.

5.2 Frequencee Domain

By use of Fourier transform and applying it in the motion equation, resulting in:

$$\int_{-\infty}^{\infty}(m\ddot{\mathrm{v}}+c\,\dot{\mathrm{v}}+k\,\mathrm{v})e^{-i\overline{\omega}t}dt=\int_{-\infty}^{\infty}\mathrm{p}(t)e^{-i\overline{\omega}t}dt \tag{20}$$

or:

$$(-\overline{\omega}^2m+i\overline{\omega}c+k)\,\mathrm{V}(\overline{\omega})=\mathrm{P}(\overline{\omega}) \tag{21}$$

with:

$$V(\overline{\omega})=\int_{+\infty}^{-\infty}v(t)e^{-i\overline{\omega}t}dt\ ;\quad P(\overline{\omega})=\int_{+\infty}^{-\infty}v(t)e^{-i\overline{\omega}t}dt \tag{22}$$

It is seen that the difference with respect to the complex loading is that now the integral boundaries are $-\infty$ and ∞. Continuing, one has:

$$\mathrm{V}(\overline{\omega})=\mathrm{H}(\overline{\omega})\,\mathrm{P}(\overline{\omega}) \tag{23}$$

where:

$$\mathrm{H}(\overline{\omega})=\frac{1}{-\overline{\omega}^2m+i\overline{\omega}c+k}=\frac{1}{k}\left[\frac{1}{(1-\beta^2)+i(2\xi\beta)}\right] \tag{24}$$

After the calculation of $\mathrm{V}(\overline{\omega})$, the response in time will be:

$$\mathrm{v}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{V}(\overline{\omega})e^{i\overline{\omega}t}d\overline{\omega}=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(\overline{\omega})P(\overline{\omega})e^{i\overline{\omega}t}d\overline{\omega} \tag{25}$$

6. Systems with multiple degrees of freedom

In systems with multiple degrees of freedom, the motion equations governing the displacements of the multiple parts of the structure correspond, in truth, to a matrix equation equivalent to the motion equation of systems with one degree of freedom. This matrix equation is given by:

$$\mathbf{m}\ddot{\mathbf{v}}(t)+\mathbf{c}\dot{\mathbf{v}}(t)+\mathbf{k}\mathbf{v}(t)=\mathbf{p}(t) \tag{26}$$

where **m** is the mass matrix, **c** the damping matrix, **k** the stiffness matrix, all of N x N order, and $\ddot{\mathbf{v}}(t)$, $\dot{\mathbf{v}}(t)$ and $\mathbf{v}(t)$ N x 1 vectors.

The system will also have multiple natural frequencies of vibration, represented by the vector $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}=\{\omega_1,\omega_2,\omega_3,\ldots\omega_N\} \tag{27}$$

These frequencies are the eigenvalues of a system of linear equations whose characteristic equation is:

$$\det(\mathbf{k}-\omega^2\mathbf{m})=0 \tag{28}$$

The corresponding eigenvectors are the vibration modes:

$$\{\hat{\mathbf{v}}_1,\hat{\mathbf{v}}_2,\hat{\mathbf{v}}_3,\ldots\hat{\mathbf{v}}_N\} \tag{29}$$

These eigenvectors have the following properties for $r \neq s$:

$$\hat{\mathbf{v}}_\mathrm{r}^\mathrm{T}\mathbf{m}\hat{\mathbf{v}}_s=0\,;\quad \hat{\mathbf{v}}_\mathrm{r}^\mathrm{T}\mathbf{k}\hat{\mathbf{v}}_\mathrm{s}=0 \tag{30}$$

Both properties are in fact applications of the generalized definition of inner product between $\hat{\mathbf{v}}_r$ and $\hat{\mathbf{v}}_s$. Therefore, it is said that different modes of vibration are perpendicular to each other, regarding to the mass matrix and the stiffness matrix.

Moreover, for the same vibration mode, one has:

$$\hat{\mathbf{v}}_\mathrm{n}^\mathrm{T}\mathbf{m}\hat{\mathbf{v}}_\mathrm{n}=\mathrm{M}_\mathrm{n}\,;\quad \hat{\mathbf{v}}_\mathrm{n}^\mathrm{T}\mathbf{k}\hat{\mathbf{v}}_\mathrm{n}=\mathrm{K}_\mathrm{n} \tag{31}$$

The values M_n and K_n are called, respectively, generalized mass and generalized stiffness.

When defining the set of modal vectors by the expression $\boldsymbol{\varphi}_n=\dfrac{\hat{\mathbf{v}}_n}{\sqrt{M_n}}$, one says that they are orthonormalized regarding the mass matrix, because besides that, one has that $\boldsymbol{\varphi}_n^T\mathbf{m}\boldsymbol{\varphi}_n=1$.

Using those modal vectors like columns of a square matrix, the modal matrix $\boldsymbol{\Phi}$ is formed and is given by:

$$\boldsymbol{\Phi}=\{\boldsymbol{\varphi}_1\quad \boldsymbol{\varphi}_2\quad \boldsymbol{\varphi}_3\quad \ldots\quad \boldsymbol{\varphi}_N\} \tag{32}$$

6.1 Decoupling the Motion Equations by Modal Superposition Method

The modal vectors are defined from the system eigenvectors, and therefore, are linearly independent. This fact allows use them to generate any vector **v** by linear combination:

$$\mathbf{v} = \boldsymbol{\varphi}_1 Y_1 + \boldsymbol{\varphi}_2 Y_2 + \boldsymbol{\varphi}_3 Y_3 + \cdots\cdots + \boldsymbol{\varphi}_N Y_N = \sum_{\mathrm{n}=1}^{\mathrm{N}} \boldsymbol{\varphi}_N Y_N \text{ or, in matrix form } \mathbf{v} = \mathbf{\Phi Y} \tag{33}$$

where **Y** is a vector that contains the coefficients Yn, that are called modal coordinates. Making the substitution of **v,** as defined in equation (35), in the equation of motion for systems with multiple degrees of freedom, becomes:

$$\mathbf{m\Phi\ddot{Y}}(t) + \mathbf{c\Phi\dot{Y}}(t) + \mathbf{k\Phi\, Y}(t) = \mathbf{p}(t) \tag{34}$$

that pre-multiplied by $\boldsymbol{\varphi}_n$, becomes:

$$\boldsymbol{\varphi}_n \mathbf{m\Phi\ddot{Y}}(t) + \boldsymbol{\varphi}_n \mathbf{c\Phi\dot{Y}}(t) + \boldsymbol{\varphi}_n \mathbf{k\Phi\, Y}(t) = \boldsymbol{\varphi}_n \mathbf{p}(t) \tag{35}$$

Due to the properties of orthogonality, already presented, in relation to the mass and stiffness matrices, the previous expression can be written as:

$$M_n \ddot{Y}_n(t) + C_n \dot{Y}_n(t) + K_n Y_n(t) = p_n(t) \tag{36}$$

Since **c** is assumed as proportional to the mass and stiffness matrices (Rayleigh damping).

Thus, the system of equations becomes uncoupled and the structural system of N degrees of freedom with N equations, in physical coordinates, is transformed into N equations of one degree of freedom, in modal coordinates (CLOUGH and PENZIEN, 1993). When one finds the solution of each equation of one degree of freedom (i.e., each Yn), the general solution of the system in physical coordinates becomes:

$$\mathbf{v} = \sum_{n=1}^{N} \boldsymbol{\varphi}_n Y_n \tag{37}$$

Time Domain Solution

The solution of each of the N equations in modal coordinates, when done in time domain through a convolution integral is:

$$Y_n = \frac{1}{M_n \omega_n} \int_0^t p_n(\tau) e^{-\xi_n \omega_n (t-\tau)} \operatorname{sen} \omega_{Dn}(t-\tau) d\tau \tag{38}$$

Frequency Domain Solution

Applying the Fourier transform to each of equations in modal coordinates, these are passed to the frequency domain, resulting in:

$$V_n(\varpi) = H_n(\varpi) P_n(\varpi) \tag{39}$$

where:

$$
\begin{aligned}
H_n(\overline{\omega}) &= \frac{1}{\omega_n^2 M_n}\left[\frac{1}{(1-\beta_n{}^2)+i(2\xi_n\beta_n)}\right]= \\
&= \frac{1}{\omega_n^2 M_n}\left[\frac{(1-\beta_n{}^2)-i(2\xi_n\beta_n)}{(1-\beta_n{}^2)^2+(2\xi_n\beta_n)^2}\right] \\
\mathrm{P}_n(\overline{\omega}) &= \int_{-\infty}^{\infty}\mathrm{p}_n(t)e^{-i\overline{\omega}t}dt
\end{aligned} \tag{40}
$$

Finally, applying the inverse Fourier transform, each final response becomes:

$$
Y_n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{V}_n(\overline{\omega})e^{i\overline{\omega}t}d\overline{\omega} \tag{41}
$$

8. Final considerations

This article presents a comparison between the procedures of solving the equations of motion in time and frequency domain for structural systems subjected to dynamic loads. The main purpose was pedagogic, to present the solutions of the equations of motion in an interesting sequence, first presenting them in the time domain, and after in the frequency domain. The approach shows the advantages of solving the equations in the frequency domain of a very clear way, because the differential equations become algebraic and their solutions are obtained with simple numerical operations, with the only difference that involve complex numbers, which can be handled the same way that the real numbers.

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