Order-of-Magnitude Analysis for the Approximate Solution of Thermal Sciences Problems

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Abstract ¾ The purpose of this paper is to show the merits of using the order-of-magnitude analysis in engineering education to solve engineering and science problems, e.g., taken from the fields of heat transfer and fluid mechanics. By this approach, the engineering student obtains an approximate solution to a problem identifying the dominating parameters and their influence on the described system. The solution is generally an outcome of simple algebraic operations. Four problems are considered to illustrate the power of the order-of-magnitude analysis method. In each problem, the results are discussed in respect to the implied assumptions, simplifications and range of validity. Analytical solutions are provided to assess the validity of the order-of-magnitude results.

Index Terms – Approximate solutions, order-of-magnitude analysis, scaling analysis, thermal sciences.

INTRODUCTION

The premise of the paper is this: In most natural processes sought to be studied by theoretical models, the requirement that all relevant terms of the conservation equations have to be of equal order of magnitude suggests reference scales for time, length and mass, and thereby for forces and fluxes. Once this point is noted, the functional form of the solution can be deduced without ever having to solve the mathematical problem. Order-of-magnitude or scaling analysis has been aptly characterized as "the premier method for obtaining the most information per unit of intellectual effort" [1] and it represents a good starting-point for further extensive analysis. A question may arise, whether the quantity of interest has to be measured in millimeters or meters. Said differently, on a logarithmic scale, is the value of a parameter of the order of 10^0 (i.e., approximately in the range from 0.5 to 5) or of the order of 10^3 (i.e., roughly in the range from 500 to 5,000). If the rules associated with this method are applied thoroughly, the desired functional relation among the variables and properties can be determined within a factor of order one compared to the exact result. The exact value of the proportionality coefficient cannot be affirmed by the order-of-magnitude analysis (OMA). While it may be 47 or 34, for instance, both are of the order of 10^1, it can be stated that the value will neither be 0.87 nor 356. Determination of the exact value of the proportionality constant requires the complete solution.

In many situations, the result of this analysis gives an appreciable if not sufficient knowledge of the range within which the driving forces interact in the described system. The outcome can at least be considered noticeable, when taking into account, that an "exact analysis" can a priori only be exact in respect to how the observer describes the issue.

The following rules are formulated for the Order-of-Magnitude Analysis:

- **First Rule:** The region in which the analysis is performed must always be defined in its spatial extent. In the first example below, the size of the dimension of interest is L, the plate's length.
- **Second Rule:** Any equation expresses equivalence between the order of magnitude of the two dominant terms. In more complex equations it is essential to find and equate only the important terms.
- **Third Rule:** If in a sum C of the two terms A and B, A's order of magnitude is greater than B's, then the order of magnitude of the sum C is equal to A's.
- **Fourth Rule:** If in a sum C of the two terms A and B, A's and B's orders of magnitude are equal, then the order of magnitude of the sum C is equal to either A's or B's.
- **Fifth Rule:** In any product $P = AB$ the order of magnitude of the product P is equal to the product of the orders of magnitude of the factors A and B. For a ratio $R = A/B$ the order of magnitude of the ratio R is equal to the ratio of the orders of magnitude of the numerator A and denominator B.

Notation: The symbol ˜ means equal in the order of magnitude.

SELECTED PROBLEMS SOLVED WITH THE OMA METHOD

Thickness of a Laminar Boundary Layer

This classical problem, first solved by L. Prandtl´s scholar H. Blasius in 1908 [2], will be addressed and solved by performing an order-of-magnitude analysis with respect to the Navier-Stokes and continuity equations. A similar approach of estimating the boundary layer thickness has first been undertaken by H. Schlichting [3].

In an incompressible and two-dimensional flow the conservation of mass can be expressed by the continuity equation Eq. (1):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (1)$$

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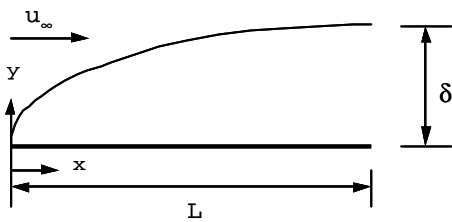
FIGURE 1
BOUNDARY LAYER OVER A SUBMERGED FLAT PLATE

If a viscous fluid flows with the free-stream velocity u_∞ along the *x*-direction over a flat plate with the length *L*, a boundary layer will develop as shown in Figure 1. In this layer the fluid is decelerated in the *x*-direction but will therefore gain a velocity component *v* in the *y*-direction. The change of the velocity in the *x*-direction *Du*, can only be of the order of the free-stream velocity u_∞, whereas the change in *x*-direction *Dx* can only be of the order of the plate length *L*. In the boundary layer the perpendicular velocity component is defined to reach from zero near the plate surface to the maximum value v_d at the edge of the boundary layer, the range in which *Dv* varies. The boundary layer itself has the thickness *d*, the maximum value for *Dy*. From mere observation we know, that the boundary layer thickness *d* is always very small compared to the dimensions of the submerged body. So, from the above considerations we can write

$$\frac{u_\infty}{L} + \frac{v_d}{d} \approx 0.$$

This allows us to estimate the perpendicular velocity in the boundary layer,

$$v_d \approx u_\infty \frac{d}{L}, \qquad (2)$$

to be of the order of the small but determinable fraction *d/L* of the free-stream velocity u_∞. The time and effort required to obtain this result (Eq. (2)) is significantly smaller when compared with the labor necessary to solve for the velocity distribution in the boundary layer and then for v_d.

Further, if the Navier-Stokes equation is simplified by assuming two-dimensional, steady-state, and isobaric flow, it may be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = x\frac{\partial^2 u}{\partial y^2},$$

where ξ is the kinematic viscosity. The individual terms can be replaced by the values they reach in the order-of-magnitude

$$u_\infty \frac{u_\infty}{L} + v_d\frac{u_\infty}{d} = x\frac{u_\infty}{d^2}.$$

If now v_d is substituted by Eq.(2)

$$u_\infty \frac{u_\infty}{L} + u_\infty \frac{d}{L}\frac{u_\infty}{d} \approx x\frac{u_\infty}{d^2},$$

the boundary layer thickness can be found by algebraic steps

$$\frac{u_\infty^2}{L} + \frac{u_\infty^2}{L} \approx x\frac{u_\infty}{d^2},$$

to be approximately

$$\frac{d}{L} \approx \sqrt{\frac{x}{u_\infty L}}.$$

Considering the definition of the Reynolds number $\text{Re}_x = ux/\xi$, the final result is obtained

$$\frac{d}{L} \approx \frac{1}{\sqrt{\text{Re}_L}}. \qquad (3)$$

Results: The thickness of the boundary layer decreases with an increase in the Reynolds number. More specific, the boundary layer thickness increases with the viscosity of the medium and decreases with the free-stream velocity of the flow. The exact solution is given by Eckert and Drake [4] to be

$$\frac{d}{L} \approx \frac{4.64}{\sqrt{\text{Re}_L}}.$$

The approximate result of Eq.(3) is found to be correct in the order of magnitude.

Laminar Flame Propagation

Assume a combustible mixture of gaseous fuel and oxidant in a long duct such as a cylindrical tube. Upon ignition of this mixture at one end of the tube, a propagating flame front will occur (Figure 2). Under steady propagation conditions, let δ be the thickness of the flame. The combustion reaction takes place within this thickness. As all reactants in the volume element of cross-section *A* times flame thickness δ are consumed, the flame will propagate into the premixed fuel.
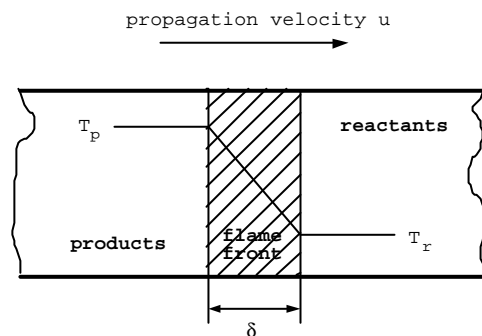

FIGURE 2
LAMINAR FLAME FRONT PROPAGATING THROUGH A DUCT.

The questions of concern here are: how thick is the flame front and what is the velocity *u* at which the flame will steadily propagate? In order to identify the primary factors, a mechanism has to be formulated and certain simplifications introduced.

- Let the tube be insulated. This assumption is easy to satisfy. The propagation is generally rapid so that heat

conduction processes into and through the tube-wall can be ignored.

- Heat transfer by means of radiation is considered negligible.
- All physical properties involved such as density ρ, specific isobaric heat c_p and mixture thermal conductivity k are assumed to remain constant over the observed range of temperature and location.

Let the entire energy release due to reaction be denoted by the source term s [W/m^3]. The heat released by the combustion reaction in the observed flame volume is therefore

$$\dot{Q}_{source} = sAd \tag{4}$$

Heat transfer from the hot products to the cold reactants will take place by conduction. Across the flame front thickness, the temperature gradient is assumed to be of the order of the difference between the product and reactant temperatures over the flame thickness d:

$$\frac{\partial T}{\partial x} \approx \frac{T_{products} - T_{reactants}}{d}$$

In conjunction with the thermal conductivity k of the mixture yields the heat flux density $\dot{Q}_{conduction} = k(T_p - T_r)/d$ over the cross-sectional area A, thus

$$\dot{Q}_{conductive} = kA\frac{T_{products} - T_{reactants}}{d} \tag{5}$$

The conducted energy will heat the gas flow by convection. This convective part can be described as follows: Let the coordinate system be fixed on the flame, which steadily moves with the unknown velocity u. Then, the reactant mass flow rate into the flame is $\dot{m} = ruA$, where r is density. This flow rate with an average specific heat c_p is heated from the supply temperature T_r to flame temperature T_p across the flame thickness d

$$\dot{Q}_{convective} = \dot{m}c_p\left(T_p - T_r\right) = ruAc_p\left(T_p - T_r\right). \tag{6}$$

The order-of-magnitude approach is based on the assumption that the foregoing three terms (Eqs. (4), (5), and (6) are of the same magnitude to allow the determination of the basic relationships between the parameters. Setting first the conductive and convective heat transfer terms equal in the order of magnitude,

$$kA\frac{T_{products} - T_{reactants}}{d} \approx ruAc_p\left(T_p - T_r\right),$$

yields

$$u \approx \frac{1}{d}\frac{k}{rc_p}.$$

The term $[k/rc_p]$ is given the symbol α and known as *thermal diffusivity* describing the ease with which temperature disturbances can be transported from one location to another in a medium. Thus,

$$u \approx \frac{a}{d}. \tag{7}$$

The conclusion that can be drawn from Eq. (7) is that thin flames propagate faster than thick ones. Next, if the source term is equated with the conduction term

$$kA\frac{T_{products} - T_{reactants}}{d} \approx sAd,$$

so that

$$d \approx \sqrt{\frac{k\left(T_p - T_r\right)}{s}}. \tag{8}$$

Eq. (8) shows that for a given k, the flame thickness primarily depends on the source strength s and the temperature difference (T_p - T_r), which are dependent parameters. Consequently, nothing else can be deduced from the last equation. But if Eq. (8) is implemented in Eq. (7), a new expression for u can be found and the final relations are

$$u = \frac{1}{rc_p}\sqrt{\frac{ks}{T_p - T_r}} \tag{9}$$

$$\left.\frac{\partial T}{\partial x}\right|_{max} = \frac{T_p - T_r}{d} = \sqrt{\frac{s\left(T_p - T\right)}{k}}. \tag{10}$$

Results: The flame propagation speed can be estimated from the properties k, r and c_p and the reaction-specific source term s. The reactant temperature T_r is known from ambient conditions, the product temperature T_p can be approximated by the adiabatic flame temperature, a quantity determined solely by thermodynamics. Preheating the mixture, low specific heat and/or density, a high thermal conductivity, and a high enthalpy of reaction $DH°$ will increase the flame speed. The derivation indicates that r and c_p influence u stronger as they appear with the power equal to unity in the equation, whereas k, s and DT come into play with the power of 1/2. Assuming a constant thermal conductivity k as well as constant product and reactant temperatures T_p and T_r, an increase in the source strength s will lead to a lower flame front thickness d and hence a stronger maximum temperature gradient. These results, agree well with complete solutions of the flame propagation problem first obtained by Landau and Lifshitz [5].

Terminal Velocity of a Sphere Falling in a Viscous Fluid

An arbitrarily shaped body falling in a viscous fluid will accelerate due to gravitational attraction. Any force exerted on a body is according to Newton's Law equal to the body's mass $m = rV$ times acceleration $a = du/dt$

$$F = ma = rV\frac{du}{dt}$$

where u is the velocity of the body. By the time it reaches a constant velocity, called the terminal velocity u_t, the problem has attained steady-state and the contributing forces - gravity, drag, and buoyancy - can be recognized as

Gravity: $F_G = -mg = -rVg$

Buoyancy: $F_B = gV(r - r_f)$

Drag: $F_D = \dfrac{1}{2} r_f C_D A u_t^2$

where the variables have the following meaning

V: volume of the body

g: gravitational acceleration constant

r: density of the body

r_f: density of the surrounding fluid

A: projected body area (frontal)

C_D: drag coefficient

According to Newton's Law, the sum of the forces must vanish at any point in time, also during acceleration.

$$\sum_i F_i = 0$$

$$F_G = F_B + F_D$$

$$rV \frac{du}{dt} = gV(r - r_f) + \frac{1}{2} r_f C_D A u_t^2 \qquad (11)$$

If this differential equation becomes one of steady-state, the transient term $rV(du/dt)$ vanishes, the terminal velocity u_t is reached, and the remaining force terms can be equated in the order of magnitude

$$gV(r - r_f) \approx \frac{1}{2} r_f C_D A u_t^2$$

to obtain an expression for the terminal velocity

$$u_t^2 \approx \frac{2gV}{C_D A}\left(\frac{r}{r_f} - 1\right). \qquad (12)$$

The time to reach $u = u_t$ is gained by equating the order of magnitude of the term on the left-hand side of the differential Eq. (11) with the order of magnitude of the buoyancy term. Assuming that

$$\frac{du}{dt} \approx \frac{\Delta u}{\Delta t} \approx \frac{u_t}{t}$$

yields

$$rV \frac{u_t}{t} \approx gV(r - r_f) \Rightarrow t \approx \frac{u_t}{g} \frac{1}{1 - \dfrac{r_f}{r}}.$$

If the relation for u_t (Eq.(12)) is implemented, one obtains

$$t \approx \sqrt{\frac{2V}{gAC_D}\left(\frac{1}{1 - \dfrac{r_f}{r}}\right)} \qquad (13)$$

For simplicity, the body is taken to be a sphere with radius R, so the volume is $V = (4/3)pR^3$ and the frontal area $A = pR^2$. For this case the terminal velocity and the time to reach it can be calculated:

$$u_t \approx \sqrt{\frac{8gR}{3C_D}\left(\frac{r}{r_f} - 1\right)} \text{ and } t \approx \sqrt{\frac{8R}{3gC_D}\left(\frac{1}{1 - \dfrac{r_f}{r}}\right)}. \qquad (14)$$

Results: For the case the drag coefficient is considered constant, one can deduce from Eq.(14)

- Large spheres have a higher terminal velocity than small ones since the gravitational force is proportional to the volume ($\propto R^3$) and the drag is proportional to the projected area ($\propto R^2$).
- The higher the body's density r in respect to the fluid's density r_f, the higher u_t.
- If the sphere's density is much larger than the fluid's density, the time to reach the terminal velocity increases with the radius R only. The smaller the difference in densities, the longer it takes for the sphere to reach the final velocity providing a constant radius.
- The qualitative drag coefficient C_D stands for the viscosity-induced skin friction drag in addition to form drag due to flow separation. The more viscous the fluid is, the lower the terminal velocity will be as well as the time to reach it.

Comparison with analytical approach: For incompressible flow, the Navier-Stokes equation of motion may be written as

$$r \frac{\partial u}{\partial t} + ru \cdot \nabla u = -\text{grad } p + m\nabla^2 u$$

while for reasonably low velocities and steady-state condition it may be written as

$$m\nabla^2 u = \text{grad } p$$

Stokes' Solution can be obtained when expressing this equation in spherical coordinates and applying the zero-slip boundary condition on the body surface. The total drag is then found to be $F_D = 6pmRu_t$. This relation is used to determine the drag coefficient by

$$C_D = \frac{F_D}{\dfrac{1}{2} r_f A u_t^2} = \frac{6pmRu_t}{\dfrac{1}{2} r_f p R^2 u_t^2} = \frac{12m}{r_f R u_t} = \frac{24}{\text{Re}}$$

and used in Eq.(14)

$$u_t \approx \frac{2gR^2}{9m}(r - r_f) \text{ and } t \approx \sqrt{\frac{2R^2 r_f}{9gm}\left(\frac{1}{1 - \dfrac{r_f}{r}}\right)}. \qquad (15)$$

Comment: Even though the powers for the parameters in Eq.(15) are different from the ones in the order-of-magnitude analysis (Eq.(14)), all the conclusions drawn from the order-of-magnitude analysis still hold when compared with the more extensive analytical derivations.

Penetration Depth of Disturbances

Time-dependent transfer processes are encountered in various forms. How deep into the soil will freezing temperatures penetrate when frosty winter weather surprises the delicate vegetation of spring? How long will it take to smell the perfume of the lady sitting down three rows away from you in the movie theater?

The way diffusion of temperature, species and momentum occur is illustrated by a microscopic view through what is known as the random walk problem, to arrive at relations for the collective behavior of gases. This issue was first addressed and solved by A. Einstein [6] and is known as the *drunkard's walk*. It is desired to know, how far an object will move away from a reference point in a given time interval supposing it always changes its direction randomly after a constant path length just as a drunkard would stagger around a lamppost. Kinetic theory of gases concerns itself with the behavior of a single, idealized particle in a large ensemble of alike particles under thermal (Brownian) motion. The path length becomes the mean free path length l after which on average a collision with a fellow particle takes place and the final distance to the starting point is called average displacements.

Two important assumptions are employed:
- While one particle is observed, all others remain fixed.
- All particles are identical and behave like perfectly elastic spheres (billiard ball model).

A particle moving a time interval Dt with velocity u and diameter s sweeps out a cylindrical volume $DV = ps^2 uDt$. Since all other molecules are stationary in this idealized set-up, all molecules in this volume element will be struck by the observed one. There being n particles per unit volume, the number of collisions in the time span Dt is $nps^2 uDt$. The collision frequency is the number of collisions divided by the elapsed time

$$Z = nps^2 u. \tag{16}$$

The number of steps taken in a given period of time is therefore

$$N = Zt. \tag{17}$$

If the mean free path length l is the distance traveled by the particle on an average between two successive collisions, it can be expressed as the distance traveled per unit time Dt divided by the number of collisions experienced in the same time

$$l = \frac{u\Delta t}{pns^2 u\Delta t} = \frac{1}{pns^2}. \tag{18}$$

With these relations derived, the drunkard's walk problem can be approached. Illustrated by Figure 3, the drunkard steps in a straight line for a fixed distance and then randomly chooses a new direction and again steps in a straight line for a fixed distance. In order to simplify this problem, it is treated in the two dimensions x and y. After N steps the mean displacement s from the origin is according to Pythagoras´ Law
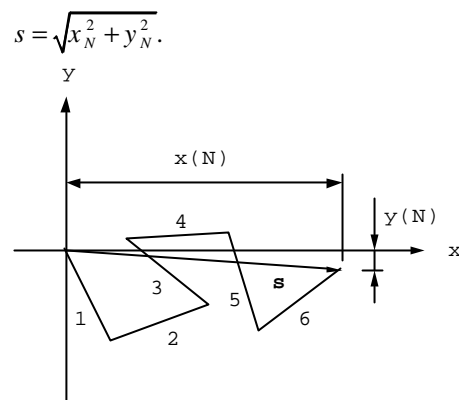
$$s = \sqrt{x_N^2 + y_N^2}.$$

FIGURE 3
PATH OF A PARTICLE UNDER RANDOM MOTION.

Each component x_N, y_N of the displacement s is the algebraic sum of the corresponding components of the previous single steps

$$x_N = x_1 + x_2 + \text{L} + x_N = \sum_{i=1}^{N} x_i$$

$$y_N = y_1 + y_2 + \text{L} + y_N = \sum_{i=1}^{N} y_i$$

$$s^2 = \left(\sum_{i=1}^{N} x_i\right)^2 + \left(\sum_{i=1}^{N} y_i\right)^2$$

$$s^2 = \sum_{i=1}^{N}\left(x_i^2 + y_i^2\right) +$$
$$2\left[\left(\sum_{i=1}^{N-1} x_i x_{i+1} + x_N x_1\right) + \left(\sum_{i=1}^{N-1} y_i y_{i+1} + y_N y_1\right)\right]$$

In each coordinate there will be as many positively as negatively directed moves, so that on the average all mixed terms will be cancelled out. What remains is

$$s^2 = \sum_{i=1}^{N}\left(x_i^2 + y_i^2\right).$$

If the steps taken are equal to the constant mean free path length l, it must hold that

$$l^2 = x_i^2 + y_i^2.$$

The mean displacement can now be stated in terms of the number of steps taken and the mean free path length l

$$s = l\sqrt{N}.$$

Eqs. (16) to (18) are now introduced and we gain

$$s = \frac{1}{pns^2}\sqrt{pns^2 ut} = \sqrt{\frac{ut}{pns^2}}$$

Results: The mean displacement from a given datum point increases with the square root of the elapsed time. From the preceding derivation one can expect that any kind of diffusion process, whether this is heat, mass, or momentum transfer, reveals this $t^{1/2}$- behavior.

To verify this statement, an example is chosen, which deals with a transient momentum transfer process. A horizontal plane wall with an infinitely thick layer of a viscous

fluid resting on it, is suddenly accelerated from rest. This configuration, known as Stokes´ First Problem, is described by the simplified one-dimensional Navier-Stokes Equation

$$\frac{\partial u}{\partial t} = n \frac{\partial^2 u}{\partial y^2}. \tag{19}$$

Comment: For a one-dimensional unsteady heat conduction problem of a cold body suddenly heated on one side, this equation would look like

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 \left(r c_p T \right)}{\partial y^2}$$

where a is the thermal diffusivity, r the fluid's density and c_p the isobaric specific heat. For the case of unsteady diffusion of species (mass transfer) we can write

$$\frac{\partial C_i}{\partial t} = D \frac{\partial^2 C_i}{\partial y^2}$$

where D is the coefficient of diffusion and C_i the concentration of species i. The similarity of the transport phenomena of heat, concentration and momentum is illustrated and emphasized by showing the similarity of their mathematical descriptions.

Going back to the plane wall problem, it is of interest how the depth of penetration of the perturbed fluid will change with time. The principles of the OMA allow us to replace the differentials with differences, so that Eq. (19) becomes

$$\frac{\Delta u}{\Delta t} = n \frac{\Delta u}{\left(\Delta y \right)^2}$$

and we can solve for the penetration depth with $Dt = t$

$$\Delta y = \sqrt{n\, t} \tag{20}$$

In viscous fluids momentum is transferred by the means of diffusion, i.e., the fluid dissipates energy by allowing particles to transfer parts of their momentum or kinetic energy, respectively, in directions other than the general flow direction. The thickness of the layer, which has been affected by this process, grows steadily with the square root of time.

Comparison with analytical approach: The flat plate problem is defined with the following boundary conditions

$$t = 0 \quad u(y) = 0$$
$$t > 0 \quad u(y = 0) = u_0$$
$$u(y = \infty) = 0,$$

and the pressure is assumed to be constant in the fluid. The substitution

$$h = \frac{y}{2\sqrt{n\, t}}$$

and assumption that $u = u_0 f(h)$, leads to the ordinary differential equation for $f(h)$

$$f'' + 2h f' = 0$$

The solution $u = u_0 \mathrm{erfc}(h)$ includes the complementary error function

$$\mathrm{erfc}(h) = \frac{2}{\sqrt{p}} \int_h^\infty e^{-h^2}\, dh,$$

which can be found tabulated in any book of mathematical tables. At $h = 2.0$ this function has a value of about 0.01, so that at $h = 2.0$ the fluid has been accelerated to 1% of the plate velocity. If this is accepted as reasonable value to assess how deep the disturbance has propagated into the fluid, we can define this distance $y(u = 0.01 u_0) = d$, the boundary layer thickness. From the substitution we can derive the final result

$$d = 2 h_d \sqrt{n\, t} \cong 4\sqrt{n\, t} \tag{21}$$

Results: The analytical solution (Eq.(21)) and the one found by OMA (Eq.(20)) applied to the simplified Navier-Stokes equation, as well as the microscopic view on a single particle in the random walk analysis, show a high consistency in describing diffusion transport mechanisms. Disturbances, whether in respect to temperature, concentration, or momentum, are shown to migrate with the square root of the elapsed time into the observed material.

CONCLUSIONS

With four problems described, an engineering education methodology is illustrated, which incorporates the distinct advantage of roughly outlining the behavior of the variable/s with a minimum input of time and/or mathematical complexity. And yet, the engineering student gains considerable knowledge by the order-of-magnitude analysis. The OMA is more sophisticated than dimensional analysis in that it not only suggests the dimensionless groups, but also a functionality of their interdependence. What it cannot do, is determining the value of the proportionality constant(s) in this functionality. Exact solutions are required to do this determination with the attended costs of time and effort. It is the scientist, or engineer who decides what degree of accuracy is sufficient and the model can be refined from there on gradually.

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