

An Iterative Calculus Programming Application

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Abstract: Mathematics subjects have been exposed to important structural changes at the Escuela Universitaria de Ingeniería Técnica Industrial de Valencia within an integrated multidisciplinary design supported by Universidad Politécnica de Valencia.

Computer aided classrooms, shared by all subjects, are nowadays commonly used. This has extended the capability of using mathematical software within the theoretical expositions which have become much more practical and technological orientated, as well as facilitating the active participation of the student and reducing the time taken up to tedious calculation tasks.

The aim of this work is to present how different mathematical software commonly used in engineering courses handle the calculation of roots of a complex number and how it can be improved by use of an adequate iterative process based on the rotation of sets in the complex field when multiplied by the exponential of an imaginary number.

Keywords: Roots of a complex number, iterative process, DERIVE, MATHEMATICA, MAPLE.

1. Introduction

Computer aided classrooms are nowadays commonly used in Mathematics classes. This has arrived further than just spending some time in a lab classroom each week at the Escuela Universitaria de Ingeniería Técnica Industrial in Universidad Politécnica de Valencia and computers form an integral part of the class.

In this note we address a mathematical issue which is commonly used in Engineering courses, above all in Electrical and Electronic Engineering, such as calculating the roots of a complex number. This is a basic task and although we teach our students that there are n different complex numbers which are the n -th root of a given complex number, if no precaution is taken, computers usually provide us with just one of them.

The aim of this note is to present how different mathematical software commonly used in engineering courses handle the calculation of roots of a complex number and how it can be improved by use of an adequate iterative process based on the rotation of sets in the complex field when multiplied by the exponential of an imaginary number.

2. Standard procedure to finding the roots of a complex number

We start by exposing how the program DERIVE version DOS deals with the calculus of roots of a complex number [1]. The WINDOWS version deals in a completely parallel way.

A complex number z may be represented in the coordinate plane by a point P which allows to characterize z by its modulus $|z|$, that is the distance from the origin to P , and its argument $\arg(z)$, that is the angle of P from the x-axis. It is standard to restrict the argument in radians to the interval $]-\pi, \pi]$, to the so called principal argument. If z is a point in the complex plane and n is a natural number, then $z^{1/n}$ can take n distinct values in the complex plane, which by de Moivre's theorem are

$$|z|^{1/n} \left(\cos \left(\frac{\arg(z) + 2k\pi}{n} \right) + i \sin \left(\frac{\arg(z) + 2k\pi}{n} \right) \right) \quad k = 0, 1, \dots, n-1$$

The modulus of any of these values is equal to $|z|^{1/n}$, and the principal arguments are n equally spaced angles. For example, the modulus of $9^{1/2}$ is 3, but the argument can be either 0 radians yielding 3, or π radians yielding -3 .

When simplifying $z^{1/n}$ the DERIVE program just provides one single value according to the option selected. The principal branch of $z^{1/n}$ is the point in the complex plane whose modulus is $|z|^{1/n}$ and whose phase (which is the

word used by DERIVE for argument) is the principal phase of z divided by n . According to this notation, the the principal branch of $9^{1/2}$ is 3 and the principal branch of $(-8)^{1/3}$ is $1 + i\sqrt{3}$.

By means of the **Manage Branch** command (**Declare Algebra State Simplification Branch** in the WINDOWS version) we may change the root DERIVE returns when simplifying rational powers by choosing a different argument of the n possible ones of $z^{1/n}$. By means of this command we may choose between **Principal Real Any**.

The **Principal** branch option forces DERIVE to select this option and is most widely accepted systematic choice.

The **Real** branch option forces DERIVE to select a real root if one exists. So, with this option, $(-8)^{1/3}$ is simplified to -2 instead of $1 + i\sqrt{3}$.

The **Any** branch option allows DERIVE to use the maximum number of transformations when it does not matter which branch is chosen. In this option DERIVE applies systematically the transformations

$$(z^m)^{1/n} \rightarrow z^{m/n}, \quad (z w)^{1/n} \rightarrow z^{1/n} w^{1/n}.$$

Therefore DERIVE produces a single value of $z^{1/n}$ when there are actually n different complex numbers.

With other scientific programs such as MATHEMATICA and MAPLE, the situation is similar and just one of the possible roots is obtained [2-3]. So, for instance if we ask to simplify $(-1)^{1/4}$ with MAPLE we will get just the value

$$\left[\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right].$$

There are two standard ways to solve the above failure of obtaining all the roots of a complex number z . One of them is to ask the program to solve an equation whose solution are the n -roots of z , that is to solve the equation

$$x^n - z = 0.$$

This usually produces satisfactory results with the DERIVE program but on some occasions the results are not so good with the others when z is not real. The other possibility is to apply de Moivre's formula and obtain one by one the n possible values of $z^{1/n}$ such as may be found for instance in [4].

3. Finding the roots of a complex number as an iterative calculus application

Iterative calculus are important in Engineering courses and appear repeatedly. For this reason we think it is interesting that students learn how an easy iterative process helps to calculate all the roots of a given complex number.

We will start from the fact that students know that the n -th roots of a complex number may be represented as the vertices of a regular polygon with n sides centered in $(0,0)$. This is due to de Moivre's theorem. Hence once we have got one of the n -th roots of a complex numbers the others may be obtained by rotating $2\pi/n$ radians ($n-1$) times, which is equivalent to multiply the previous root by $e^{2\pi i/n}$.

On the other hand, the DERIVE program incorporates a command to make iterative calculations, which in fact is commonly used in numerical calculus. This command is

$$\text{ITERATES}(u(x), x, x_0, n)$$

where x_0 is the initial point, $u(x)$ is a function of x , and n is the number of times to repeat the iterative process. When we simplify the above expression the program returns

$$[x_0, u(x_0), u^2(x_0), \dots, u^{n-1}(x_0)]$$

where $u^2(x_0) = u(u(x_0))$, $u^3(x_0) = u(u(u(x_0)))$, and so on.

Therefore, taking $u(x) = x e^{2\pi i/n}$ and as initial value the root provided by the program when we simplify

$$z^{1/n}$$

the command

$$\text{ITERATES}(x \cdot e^{(2\pi i/n)}, x, z^{1/n}, n-1)$$

enables to find the n -th roots of the complex number z . In the above expression we may omit the last argument $n-1$ since in that case the program repeats the iterative process until it finds some values which coincides with some of the previously calculated ones, which is to happen in this case since after repeating the process n times we will get the initial value $z^{1/n}$.

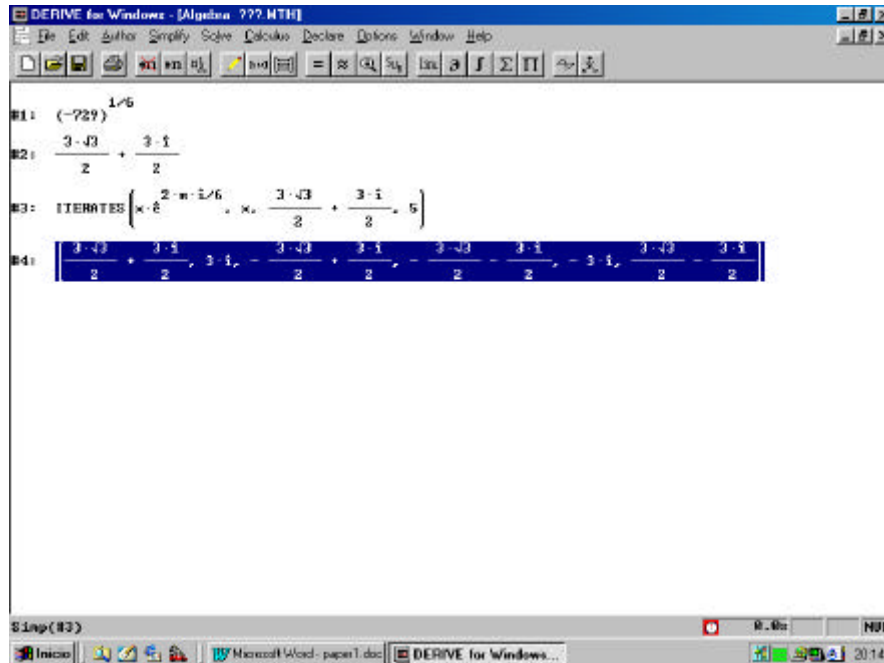
For example if we want to find the six sixth-roots of -729 we are able to gather the two previous steps and just introduce in the **Author** line

$$\text{ITERATES}(x \cdot e^{(\pi i/3)}, x, (-729)^{(1/6)}, 5)$$

Simplifying the above expression, the program will give us

$$\left[\frac{3\sqrt{3}}{2} + \frac{3i}{2}, 3i, -\frac{3\sqrt{3}}{2} + \frac{3i}{2}, -\frac{3\sqrt{3}}{2} - \frac{3i}{2}, -3i, \frac{3\sqrt{3}}{2} - \frac{3i}{2} \right].$$

In the WINDOWS version, and keeping the two separate steps we would have got the following



Finally if we had introduced

$$\text{ITERATES} \left(x \cdot e^{(pi/3)}, x, (-729)^{(1/6)} \right)$$

and simplify the above expression, the program would have provided us with

$$\left[\frac{3\sqrt{3}}{2} + \frac{3i}{2}, 3i, -\frac{3\sqrt{3}}{2} + \frac{3i}{2}, -\frac{3\sqrt{3}}{2} - \frac{3i}{2}, -3i, \frac{3\sqrt{3}}{2} - \frac{3i}{2}, \frac{3\sqrt{3}}{2} + \frac{3i}{2} \right].$$

We note that this vector has seven coordinates, the last one coinciding with the first one. The sixth roots of -729 are the first six components.

Analogously, the program MATHEMATICA also enables to obtain the six roots of this example by means of the command DO. Strictly speaking this command will not perform an iterative process but it allows to develop the same idea that is behind the above iterations by calculating, one after the other, the five rotations that the DERIVE command ITERARES generates from the only root that either of these programs generates. In fact if we ask the program MATHEMATICA to calculate

$$(-729)^{(1/6)}$$

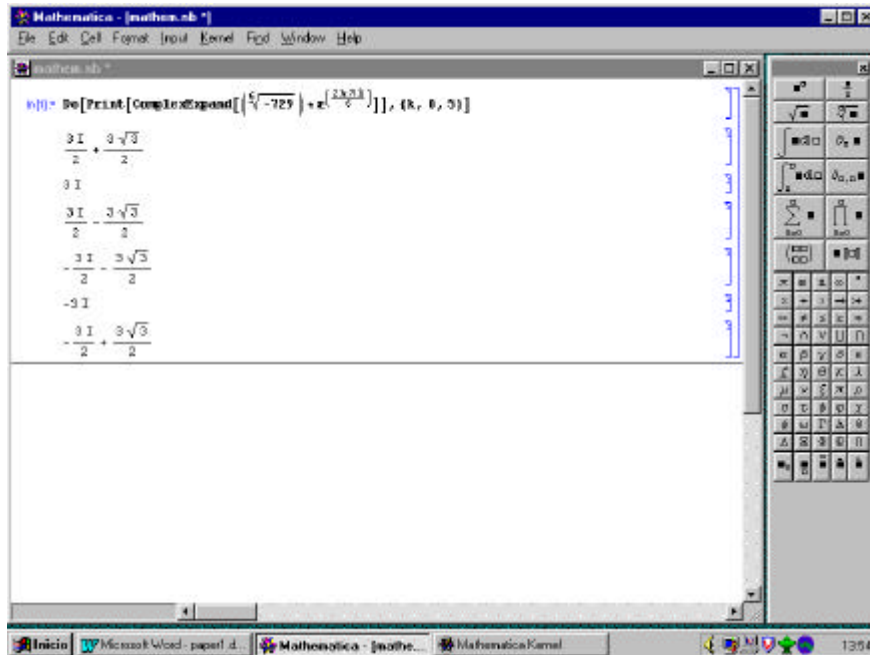
it will answer $3(-1)^{1/6}$. Not indeed a great advance.

In order to get the six roots and in the line of the process exposed we should introduce:

$$\text{Do}[\text{Print}[\text{ComplexExp and}[\% E^{(k * Pi * I/3)}]], \{k, 0, 5\}]$$

Recall that the MATHEMATICA program takes % as the last output (if we have done the previous step this symbol will be understood as $3(-1)^{1/6}$) and ComplexExpand writes a complex number in the form $x+iy$.

As with the DERIVE program we can perform these two steps in just one. In this case the screen of the computer should look like the following:



4. Conclusions

In general terms Mathematics teaching is very much helped with the use of computers. But definitely a little of the specific capabilities and way of working of the software used by the student must be known by him in order to take real advantage of it, at the same time of avoiding to get wrong conclusions.

This particular issue we have addressed (to get all the roots of a complex number) and the iterative process we suggest to find them gives us a chance not only to teach the student how to calculate all the roots with a computer but to make the student understand how we make it, by rotation of the single initial value provided by the program we are dealing with. And for this the only thing we have to do is to multiply it repeatedly by the adequate exponential of a complex number.

So the achievements are multiple: introducing the student to iterative processes, understanding de Moivre's theorem and relating rotation of α radians in the complex field with product by $e^{i\alpha}$.

5. References

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